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Time Dilation Cosmology 8:

A Formal Mathematical Architecture for Time Dilation Cosmology

Representation and Completeness Theorems

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Abstract

Time Dilation Cosmology (TDC) has been developed through a series of previous publications in which the observer-conditioned differential time-rate field is adopted as the primitive quantity from which the principal physical observables of the framework are represented. While those papers established the physical relationships underlying the theory, they did not present the formal mathematical organization of the framework itself.

The objective of the present paper is to develop a formal mathematical architecture for Time Dilation Cosmology. This architecture is constructed by introducing the primitive mathematical objects of the theory, stating the foundational assumptions as explicit axioms, deriving the Representation and Completeness Theorems through a sequence of preliminary lemmas, and illustrating the resulting framework through representative observable mappings established in the earlier TDC publication series.

The Representation Theorem demonstrates that every observable admitted by the TDC framework possesses a well-defined mathematical representation through mappings acting upon the admissible physical state. The Completeness Theorem establishes that, within the assumptions adopted by the framework, no additional primitive mathematical quantity is required for the representation of the observables admitted by the theory.

Throughout the paper, explicit distinctions are maintained between mathematical definitions, foundational assumptions, derived theorems, and physical interpretation. Consequently, the work provides a formal deductive organization of the TDC framework without rederiving the physical relationships established in the earlier publications. The resulting mathematical architecture provides a unified formal foundation for future mathematical developments and observational investigations within the Time Dilation Cosmology framework.

Keywords: Time Dilation Cosmology; Mathematical Physics; Representation Theorem; Completeness Theorem; Observer Universe; Differential Time-Rate Field; Unified Field Theory; Mathematical Cosmology.

1. Introduction

Modern theoretical physics is founded upon several highly successful mathematical frameworks, including general relativity, quantum mechanics, Hamiltonian mechanics, and relativistic cosmology. Each employs its own primitive mathematical quantities, assumptions, and methods of representation to describe different aspects of physical reality. Although these theories have achieved remarkable predictive success within their respective domains, no single primitive mathematical quantity is generally regarded as providing a common representation for gravitation, quantum phenomena, cosmological evolution, and observer-dependent measurement.

Time Dilation Cosmology (TDC), developed in the preceding papers of this publication series [1–9], is a theoretical framework in which the observer-conditioned differential time-rate field is adopted as the primitive mathematical quantity of the formalism. Within this framework, previously published work has constructed mathematical representations for gravitation, relativistic dynamics, cosmological evolution, quantum phase evolution, Hamiltonian mechanics, and observer-dependent cosmology through relationships derived from this common primitive quantity [1–9]. Those earlier publications established the physical development of the theory, whereas the formal mathematical organization of the framework has not previously been presented in a unified manner.

The objective of the present paper is therefore not to introduce additional physical laws or to rederive previously established relationships. Rather, its objective is to formalize the mathematical architecture of the Time Dilation Cosmology framework. This is accomplished by introducing the primitive mathematical objects of the theory, stating its foundational assumptions as explicit axioms, deriving Representation and Completeness Theorems through a sequence of preliminary lemmas, and demonstrating how previously established observable mappings are organized within a single deductive mathematical structure.

The mathematical development presented here distinguishes explicitly between definitions, assumptions, derived theorems, and physical interpretation. Accordingly, the Representation and Completeness Theorems establish the logical organization of the TDC framework under its stated assumptions rather than serving as independent demonstrations of its empirical validity. Questions concerning observational agreement and experimental verification remain matters of physical investigation rather than mathematical deduction.

The remainder of the paper is organized as follows. Section 2 places the TDC framework within the broader context of established mathematical physics. Section 3 introduces the primitive mathematical objects employed throughout the theory. Section 4 states the foundational axioms. Section 5 develops the preliminary lemmas required for the principal results. Sections 6 and 8 establish the Representation and Completeness Theorems, respectively, while Sections 7 and 9 discuss their scope and limitations. Appendix A summarizes representative observable mappings developed in the earlier TDC publication series and illustrates the mathematical organization established by the present work.

2. Relation to Established Mathematical Physics

The mathematical description of physical phenomena has historically been developed through several complementary theoretical frameworks. General relativity represents gravitation through the spacetime metric tensor and Einstein's field equations [10-12]. Hamiltonian mechanics represents dynamical evolution through the Hamiltonian function [13]. Quantum mechanics represents physical states and their evolution through wavefunctions, operators, and the Schrödinger equation [14-16]. Relativistic cosmology describes the large-scale evolution of the universe through the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [17-20]. Each framework therefore employs its own primitive mathematical quantities and corresponding structures.

A recurring objective in mathematical physics has been the reduction of the number of independent primitive quantities required to represent physical systems. Such reductions do not, by themselves, establish physical correctness. Rather, they may provide a more economical mathematical architecture within which physical relationships can be expressed. The present work adopts this principle as a mathematical motivation rather than as empirical evidence.

General Relativity relates spacetime curvature to the distribution of mass and energy through Einstein's field equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \qquad (1)$$

Here, $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, Λ is the cosmological constant, G is Newton's gravitational constant, and c is the invariant speed of light.

Special relativity relates proper time to coordinate time through

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt. \qquad (2)$$

Quantum evolution is represented through the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{H}\psi, \qquad (3)$$

where ψ is the wavefunction and $\widehat{H}$ is the Hamiltonian operator.

For a spatially flat homogeneous and isotropic universe, the FLRW metric is

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + r^2 d\Omega^2], \qquad (4)$$

with cosmological redshift given by

$$1 + z = \frac{a_0}{a_e}, \qquad (5)$$

where a_0 and a_e are the scale factors at observation and emission, respectively.

Time Dilation Cosmology adopts a different mathematical starting point. Within TDC, the observer-conditioned differential time-rate field is taken as the primitive quantity from which the principal observables admitted by the framework are represented. One previously established relation is

$$dR_t = \frac{v^2}{c^2}, \qquad (6)$$

from which velocity follows as the derived observable

$$v = c\sqrt{dR_t}. \qquad (7)$$

Within the cosmological formulation, the FLRW scale factor is represented by the time-rate field,

$$a(t) = R_t(t), \qquad (8)$$

so that the standard redshift relation becomes

$$1 + z = \frac{R_t(\text{obs})}{R_t(\text{emit})}. \qquad (9)$$

The differential time-rate and velocity relations were developed in the earlier TDC papers [1-3]. The gravitational formulation was extended in *The Unified Field* [4], while the

Hamiltonian, quantum-phase, Einstein-equation, and FLRW representations were organized in the later work [9].

The equations above are included to identify the established physical relationships that motivate the formal structure developed in the present paper. They are not rederived here. The objective of the present work is instead to define the mathematical objects, assumptions, mappings, and logical relationships through which these previously established results are organized.

Accordingly, the present paper distinguishes four levels of mathematical development:

1. **Definitions**, which introduce the mathematical objects employed by the framework.
2. **Fundamental assumptions or axioms**, which state the premises under which those objects are used.
3. **Theorems**, which are derived from the definitions and axioms.
4. **Physical interpretation**, which assigns physical meaning to the resulting mathematical structure.

Maintaining these distinctions separates assumptions from derived conclusions and clarifies the logical scope of the Representation and Completeness Theorems.

The Representation Theorem establishes the conditions under which observables admitted by TDC possess well-defined mappings of the differential time-rate field. The Completeness Theorem then examines whether an additional primitive mathematical quantity is required to represent those observables.

The scope of the present work is therefore restricted to the formal mathematical organization of the TDC framework. Questions concerning observational agreement, empirical verification, and the physical interpretation of the previously derived relationships remain matters of theoretical and experimental investigation.

3. Fundamental Mathematical Definitions

The purpose of this section is to define the mathematical objects employed throughout the Time Dilation Cosmology framework. These definitions establish the notation and formal structures used in the subsequent axioms, lemmas, and theorems. They do not, by themselves, constitute physical laws or empirical claims.

3.1 Mathematical and Observer Universes

Definition 1 (Mathematical Universe)

A mathematical universe is defined as the ordered triple

$$U = (X, \mathcal{O}, \mathcal{B}), \qquad (10)$$

where X is the spacetime domain, $\mathcal{O}$ is the set of admissible observers, and $\mathcal{B}$ is the collection of boundary, initial, and observational conditions required to specify an admissible configuration.

The mathematical universe provides the domain within which all subsequent fields, states, and mappings are defined.

Definition 2 (Observer Universe)

For each observer

$$O \in \mathcal{O},$$

the corresponding observer universe is denoted by

$$U_O \subseteq U. \qquad (11)$$

The observer universe U_O consists of the domain, conditions, and represented quantities associated with the specified observer O.

Different observers may possess different observer universes while remaining elements of the same mathematical universe.

3.2 Primitive Scalar Field

Definition 3 (Differential Time-Rate Field)

The observer-conditioned differential time-rate field is a scalar mapping

$$dR_t : (X \times T) \times O \to \mathbb{R}, \qquad (12)$$

written explicitly as

$$dRt = dR_t(x, t \mid O). \qquad (13)$$

Here, x denotes spatial position, t denotes the temporal coordinate, and O denotes the observer condition.

Equation (13) specifies the mathematical dependence of the field. Its status as the primitive quantity of the TDC framework is stated later as an axiom rather than assumed as part of the present definition.

3.3 Physical States

Definition 4 (Physical State)

A physical state is an admissible configuration consisting of the differential time-rate field together with the corresponding observer universe:

$$S = (dR_t, U_O). \qquad (14)$$

The collection of all admissible physical states is denoted by

$$\mathcal{S} = \{(dR_t, U_O)\}. \qquad (15)$$

Each element of $\mathcal{S}$ must satisfy the boundary, initial, and observational conditions contained in $\mathcal{B}$.

3.4 Observable Structure

Definition 5 (Observable)

An observable is a measurable mathematical quantity defined on the state space:

$$Q_i: \mathcal{S} \to \mathbb{R}. \qquad (16)$$

The index i labels the observable under consideration. Examples within the TDC framework include velocity, acceleration, energy, Hamiltonian, quantum phase, redshift, and metric quantities.

No assumption is made at this stage concerning the origin or explicit mathematical form of an observable.

Definition 6 (Observable Functional)

An observable functional is a mapping

$$\mathcal{F}_i: \mathcal{S} \to \mathbb{R} \qquad (17)$$

that assigns an observable value to each admissible physical state. Thus,

$$Q_i = \mathcal{F}_i(S). \qquad (18)$$

Using Equation (14), the same representation may be written as

$$Q_i = \mathcal{F}_i(dR_t, U_O). \qquad (19)$$

Equations (18) and (19) define the form of an observable representation. They do not establish that such a functional exists for every observable admitted by the framework. That requirement is introduced explicitly in Section 4.

Definition 7 (Representation Space)

The representation space is the collection of all admissible observable functionals:

$$\mathcal{R} = \{\mathcal{F}_i\}_{i \in I}, \qquad (20)$$

where I is the index set labeling the observables admitted by the framework.

Each element of $\mathcal{R}$ maps an admissible physical state to a represented observable value.

3.5 Classification of Variables

Definition 8 (Primitive Variable)

A primitive variable is a mathematical quantity whose value is not defined in terms of another variable within the formal structure of the theory.

Whether a particular quantity is primitive is determined by the axioms of the framework rather than by definition alone.

Definition 9 (Derived Variable)

A derived variable is a mathematical quantity obtained through a mapping of one or more primitive variables.

If P denotes the collection of primitive variables and D_j is derived, then

$$D_j = \mathcal{G}_j(P), \qquad (21)$$

for some admissible mapping $\mathcal{G}_j$.

Definition 10 (Independent Arguments of the Field)

The independent arguments of the differential time-rate field are

$$x,\ t,\ O. \qquad (22)$$

Accordingly,

$$dR_t = dR_t(x, t \mid O). \qquad (23)$$

No particular analytical form, field equation, or boundary condition is imposed by this definition.

3.6 Logical Role of the Definitions

The preceding definitions establish the mathematical vocabulary of the TDC framework. They introduce the mathematical universe, observer universes, the differential time-rate field, admissible physical states, observable functionals, and the representation space.

These definitions do not establish that dR_t is the unique primitive quantity, that every observable possesses a representation through it, or that such representations are complete. Those propositions are introduced as explicit assumptions in Section 4 and examined through the lemmas and theorems that follow.

The preceding definitions therefore specify the mathematical entities of the framework. The following section states the foundational assumptions under which those entities are employed.

4. Fundamental Assumptions (Axioms)

The preceding section defined the mathematical objects employed by the Time Dilation Cosmology framework. The present section states the assumptions under which those objects are used. These assumptions are not derived within this paper. They specify the formal structure from which the preliminary lemmas and subsequent theorems follow.

To avoid assuming the conclusions of the Representation and Completeness Theorems in advance, the axioms below do not assert that the differential time-rate field is the only possible primitive quantity in every physical theory. They define the mathematical premises adopted specifically within the TDC framework.

Axiom 1 (Primitive-Field Postulate)

The observer-conditioned differential time-rate field

$$dR_t = dR_t(x, t \mid O) \qquad (24)$$

is adopted as the primitive scalar field of the TDC formalism.

All other quantities represented within the framework are treated as either observer conditions, boundary data, or derived quantities.

This axiom identifies the mathematical starting point of the theory. It does not establish that no alternative primitive quantity could be used in another theoretical framework.

Axiom 2 (Observer-Conditioned State)

Every admissible physical state is defined relative to an observer universe U_O. Accordingly,

$$S = (dR_t, U_O). \qquad (25)$$

The field configuration cannot be separated from the observer, boundary, initial, and observational conditions under which it is defined.

Thus, two field configurations are regarded as the same admissible state only when both the field and the observer universe are identical:

$$(dR_t^{(1)} U_O^{(1)}) = (dR_t^{(2)} U_O^{(2)}). \qquad (26)$$

Axiom 3 (State Sufficiency)

For every observable admitted by the TDC framework, all mathematical information required to determine its represented value is contained in the corresponding admissible state

$$S = (dR_t, U_O). \qquad (27)$$

No additional state variable is assumed unless it is included within the observer universe or is derivable from the primitive field and the stated conditions.

This axiom does not yet specify the explicit functional form of any observable representation. It states only that the admissible state contains the information required to determine the represented observable.

Axiom 4 (Determinacy)

For every admissible state S, each observable admitted by the framework has one represented value.

Thus, if

$$S_1 = S_2, \qquad (28)$$

then, for every admitted observable Q_i,

$$Q_i(S_1) = Q_i(S_2). \qquad (29)$$

Equivalently, identical admissible states cannot produce different represented values of the same observable.

This assumption establishes single-valuedness within the formal representation. It does not assert that nature itself is deterministic beyond the mathematical domain defined by the framework.

Axiom 5 (Representation Consistency)

If two admissible mathematical expressions represent the same observable under identical state and observer conditions, they must yield the same observable value.

Let $\mathcal{F}_i$ and $\mathcal{G}_i$ be two admissible representations of the same observable Q_i. Then

$$\mathcal{F}_i(S) = \mathcal{G}_i(S) \qquad (30)$$

for every state S on which both mappings are defined.

This axiom ensures that the value assigned to an observable does not depend upon the choice of equivalent mathematical representation.

Axiom 6 (Admissibility of Observable Mappings)

A mathematical quantity is admitted as an observable of the TDC framework only if its represented value is determined by an admissible state.

Accordingly, for each admitted observable Q_i, there exists a mapping

$$\mathcal{F}_i: \mathcal{S} \to \mathbb{R} \qquad (31)$$

such that

$$Q_i = \mathcal{F}_i(S). \qquad (32)$$

Using Equation (25), this may be written as

$$Q_i = \mathcal{F}_i(dR_t, U_O). \qquad (33)$$

The explicit form of $\mathcal{F}_i$ depends upon the observable under consideration and is not supplied by the present axiom.

Axiom 7 (Domain Restriction)

The formal conclusions developed in this paper apply only to:

1. observables admitted by the TDC framework;
2. admissible states belonging to $\mathcal{S}$;
3. mappings belonging to the representation space $\mathcal{R}$; and

4. observer, boundary, and initial conditions included within the relevant observer universe.

No conclusion is asserted for observables or physical systems lying outside this defined domain.

4.1 Logical Role of the Axioms

The axioms above establish the formal premises of the TDC mathematical architecture.

Axiom 1 identifies the differential time-rate field as the primitive scalar field adopted by the framework.

Axiom 2 defines the observer-conditioned nature of every admissible state.

Axiom 3 states that the admissible state contains the information required to determine each represented observable.

Axiom 4 establishes the single-valuedness of observable representations.

Axiom 5 requires equivalent representations of the same observable to agree.

Axiom 6 states the admissibility condition for observable mappings.

Axiom 7 restricts all subsequent conclusions to the explicitly defined domain of the theory.

These assumptions do not independently establish that the TDC framework is empirically correct, physically unique, or universally applicable. They define the formal mathematical system within which the preliminary lemmas and the Representation and Completeness Theorems are developed.

The following section derives the immediate mathematical consequences of these definitions and axioms.

5. Preliminary Lemmas

The following lemmas establish the immediate mathematical consequences of the definitions and axioms introduced in the preceding sections. They do not constitute the principal results of the paper but provide the logical foundation upon which the Representation and Completeness Theorems are established.

Lemma 1 (Existence of Observable Representations)

For every observable admitted by the TDC framework, there exists at least one admissible observable mapping.

Proof

By Axiom 6, every admitted observable Q_i possesses an admissible mapping

$$Q_i = \mathcal{F}_i(S), \qquad (34)$$

where

$$\mathcal{F}_i : \mathcal{S} \to \mathbb{R}. \qquad (35)$$

Therefore, every admitted observable possesses at least one mathematical representation.

□

Lemma 2 (Uniqueness of Observable Values)

For every admissible physical state, each admitted observable possesses a unique represented value.

Proof

Let Sbe an admissible state.

By Axiom 4,

$$S_1 = S_2 \Longrightarrow Q_i(S_1) = Q_i(S_2). \qquad (36)$$

Therefore identical admissible states cannot produce different represented values for the same observable.

Hence the represented value of every admitted observable is unique.

□

Lemma 3 (Dependence upon the Admissible State)

Every admitted observable depends only upon the admissible physical state.

Proof

By Axiom 3, all information required to determine an admitted observable is contained in

$$S = (dR_t, U_O). \qquad (37)$$

By Axiom 6,

$$Q_i = \mathcal{F}_i(S). \qquad (38)$$

Therefore every admitted observable depends only upon the admissible state.

□

Lemma 4 (Observer Dependence)

Every admitted observable is conditioned by the corresponding observer universe.

Proof

By Definition 4,

$$S = (dR_t, U_O). \qquad (39)$$

Since every observable is represented through

$$Q_i = \mathcal{F}_i(S), \qquad (40)$$

and $U_O$forms part of every admissible state, every represented observable necessarily depends upon the associated observer universe.

□

Lemma 5 (Representation Consistency)

Equivalent observable mappings assign identical values to every admitted observable.

Proof

Let $\mathcal{F}_i$ and $\mathcal{G}_i$ represent the same observable.

By Axiom 5,

$$\mathcal{F}_i(S) = \mathcal{G}_i(S) \qquad (41)$$

for every admissible state.

Therefore equivalent representations produce identical observable values.

□

Lemma 6 (Closure of the Representation Space)

The representation space is closed under admissible observable mappings.

Proof

By Definition 7,

$$\mathcal{R} = \{\mathcal{F}_i\}. \qquad (42)$$

Every admitted observable mapping belongs to the representation space by construction.

Consequently, every admissible observable representation remains within $\mathcal{R}$.

□

Remark

The preceding lemmas establish the elementary structural properties required for the principal results of the paper. In particular, they show that admitted observables possess well-defined representations, that represented values are unique, that every representation depends only upon the admissible state, and that equivalent representations are mathematically consistent.

These properties provide the logical foundation for the Representation Theorem developed in the following section.

6. Representation Theorem

The preceding definitions, axioms, and lemmas establish the formal structure required to state the principal mathematical result of the present work. The Representation Theorem demonstrates that every observable admitted by the Time Dilation Cosmology framework possesses a well-defined mathematical representation through mappings acting upon the admissible physical state.

Theorem 1 (Representation Theorem)

Let

$$S = (dR_t, U_O) \qquad (43)$$

be an admissible physical state belonging to the state space $\mathcal{S}$.

Assume that

1. dR_t is the primitive scalar field adopted by the framework (Axiom 1);
2. every admissible state is observer conditioned (Axiom 2);
3. every admitted observable depends only upon the admissible state (Axiom 3);
4. observable values are uniquely determined (Axiom 4); and
5. every admitted observable possesses an admissible mapping (Axiom 6).

Then every observable admitted by the TDC framework possesses a mathematical representation of the form

$$Q_i = \mathcal{F}_i(S), \qquad (44)$$

or equivalently,

$$Q_i = \mathcal{F}_i(dR_t, U_O), \qquad (45)$$

for some admissible observable functional

$$\mathcal{F}_i: \mathcal{S} \to \mathbb{R}. \qquad (46)$$

Proof

By Lemma 1, every observable admitted by the framework possesses at least one admissible observable mapping.

By Lemma 2, the represented value of every admitted observable is unique.

By Lemma 3, every admitted observable depends only upon the admissible physical state

$$S = (dR_t, U_O). \qquad (47)$$

By Lemma 4, the observer universe forms part of every admissible state, so the representation necessarily remains observer conditioned.

Consequently, every admitted observable is represented by an admissible mapping

$$\mathcal{F}_i : \mathcal{S} \to \mathbb{R},$$

such that

$$Q_i = \mathcal{F}_i(S). \qquad (48)$$

No additional mathematical structure is required to establish the existence of the representation.

Therefore every observable admitted by the TDC framework possesses a well-defined mathematical representation through the admissible state.

$$\boxed{Q_i = \mathcal{F}_i(dR_t, U_O)} \qquad (49)$$

This completes the proof.

□

Corollary 1 (Representation by the Primitive Scalar Field)

Since every admissible state is defined by

$$S = (dR_t, U_O), \qquad (50)$$

every admitted observable is represented through the primitive differential time-rate field together with the corresponding observer universe.

Accordingly,

$$Q_i = \mathcal{F}_i(dR_t, U_O) \qquad (51)$$

for every admitted observable.

Corollary 2 (Observer Dependence of Representations)

Let

$$U_{O_1} \neq U_{O_2}. \qquad (52)$$

Then the corresponding observable mappings satisfy

$$\mathcal{F}_i(dR_tU_{O_1}) \neq \mathcal{F}_i(dR_tU_{O_2}) \qquad (53)$$

unless both observer universes determine identical admissible states.

Thus, observable representations remain explicitly observer conditioned.

Corollary 3 (Unified Representation Space)

Since every admitted observable possesses a representation

$$Q_i = \mathcal{F}_i(S), \qquad (54)$$

all admitted observable mappings belong to the common representation space

$$\mathcal{R} = \{\mathcal{F}_i\}. \qquad (55)$$

Accordingly, velocity, gravitation, Hamiltonian mechanics, quantum phase evolution, cosmological redshift, FLRW cosmology, and other observables previously developed within the TDC framework are organized within a single mathematical representation space, while retaining their individual functional forms.

Remark

The Representation Theorem establishes only the existence and formal organization of observable representations within the TDC framework. It does not determine the explicit analytical form of the mappings $\mathcal{F}_i$, nor does it establish the empirical validity of the represented observables. Those mappings were developed in the earlier TDC publications and are summarized in Appendix A.

Transition

The Representation Theorem demonstrates that every observable admitted by the framework possesses a well-defined mathematical representation. A natural question then arises: **Is any additional primitive mathematical quantity required to represent those observables?** The following section examines that question by considering the scope and limitations of the Representation Theorem before establishing the Completeness Theorem.

7. Scope and Limitations of the Representation Theorem

The Representation Theorem established in the preceding section is a statement concerning the formal mathematical organization of the Time Dilation Cosmology (TDC) framework. Its conclusions are therefore restricted to the definitions, assumptions, and admissible mathematical structures introduced in the present paper. The theorem should not be interpreted as establishing broader physical conclusions beyond that defined mathematical domain.

First, the Representation Theorem establishes only the existence of admissible mathematical representations for observables admitted by the TDC framework. It does not determine the explicit analytical form of those representations. The functional mappings

$$Q_i = \mathcal{F}_i(S) \qquad (56)$$

are asserted to exist under the stated axioms, but their explicit forms were developed in the earlier TDC publications and are not rederived here.

Second, the theorem does not establish the empirical validity of the represented observables. Whether a particular mapping accurately describes a physical phenomenon remains a question of theoretical development, experimental measurement, and observational verification rather than mathematical deduction.

Third, the theorem does not imply that every mathematical quantity appearing in other physical theories must possess a representation through the TDC framework. The theorem applies only to those observables admitted by the mathematical definitions and axioms adopted in the present work.

Likewise, the theorem does not establish that the differential time-rate field is the unique primitive quantity capable of representing physical phenomena. Other theoretical frameworks may adopt different primitive variables and develop mathematically consistent representations. The Representation Theorem establishes only that, within the TDC framework, admitted observables possess representations through the admissible physical state defined by

$$S = (dR_t U_{O_1}). \qquad (57)$$

Furthermore, the theorem does not establish the uniqueness of the observable mappings themselves. Distinct mathematical expressions may represent the same observable provided they satisfy the Representation Consistency Axiom and therefore assign identical values to every admissible state.

The significance of the Representation Theorem therefore lies in the organization of the mathematical framework rather than in the derivation of new physical laws. It establishes that the previously developed observable relationships of the TDC framework may be expressed within a common deductive mathematical structure while preserving the distinctions between definitions, assumptions, derived theorems, and physical interpretation.

Accordingly, the Representation Theorem should be understood as a statement concerning the mathematical architecture of the TDC framework rather than as an assertion regarding the completeness or ultimate physical validity of the theory. Those questions lie beyond the scope of the present theorem and remain subjects for continued theoretical development and empirical investigation.

Remark

The explicit separation between mathematical representation and physical interpretation is fundamental to the logical structure of the present paper. The Representation Theorem establishes the existence and organization of observable mappings under the stated assumptions. Questions concerning the adequacy of those mappings as descriptions of physical reality are intentionally reserved for subsequent theoretical and observational analysis.

Transition

The Representation Theorem establishes that every observable admitted by the TDC framework possesses a well-defined mathematical representation. An immediate question then arises: **Is the primitive differential time-rate field together with the observer universe sufficient to represent every admitted observable, or must additional primitive mathematical quantities be introduced?**

The following section addresses this question through the Completeness Theorem.

8. Completeness Theorem

The Representation Theorem established that every observable admitted by the TDC framework possesses a well-defined mathematical representation through the admissible physical state. The remaining question is whether the primitive mathematical structure introduced in Sections 3 and 4 is sufficient for those representations or whether additional primitive mathematical quantities are required.

The following theorem addresses that question within the formal framework defined by the preceding definitions and axioms.

Theorem 2 (Completeness Theorem)

Let

$$S = (dR_t U_{O_1}) \qquad (58)$$

be an admissible physical state belonging to the state space $\mathcal{S}$.

Assume that:

1. the differential time-rate field is the primitive scalar field of the framework, as stated in Axiom 1;
2. every admissible state is observer conditioned, as stated in Axiom 2;
3. every admitted observable depends only upon the admissible state, as stated in Axiom 3;
4. every admitted observable possesses an admissible representation, as stated in Axiom 6; and
5. the domain of applicability is restricted to the observables admitted by the TDC framework, as stated in Axiom 7.

Then no additional primitive mathematical quantity is required to represent the observables admitted by the framework.

Proof

Suppose, for contradiction, that there exists an additional primitive mathematical quantity

$$P \qquad (59)$$

that is necessary for representing at least one observable admitted by the framework.

Then there exists an observable Q_k whose representation necessarily depends upon both the admissible state S and the additional primitive quantity P. Its representation would therefore have the form

$$Q_k = \mathcal{G}_k(\mathrm{S}, \mathrm{P}), \qquad (60)$$

where $\mathcal{G}_k$ is an admissible mapping.

However, by Axiom 3, all mathematical information required to determine every admitted observable is contained in the admissible physical state

$$S = (dR_t U_O). \qquad (61)$$

Furthermore, by Axiom 6, every admitted observable possesses an admissible representation of the form

$$Q_k = \mathcal{F}_k(S). \qquad (62)$$

Substituting the definition of S gives

$$Q_k = \mathcal{F}_k(dR_t U_O). \qquad (63)$$

Therefore, P cannot be necessary for representing Q_k, because the observable already possesses a complete admissible representation through S. This contradicts the assumption that P is required.

Hence no additional primitive mathematical quantity is required to represent the observables admitted by the TDC framework.

$$\boxed{Q_i = \mathcal{F}_i(dR_t U_O)} \qquad (64)$$

This completes the proof.

□

Corollary 1 (Minimal Primitive Structure)

Within the domain defined by the TDC framework, the primitive mathematical structure consists of the admissible state

$$S = (dR_t U_O), \qquad (65)$$

together with the observer, boundary, initial, and observational conditions contained within the observer universe.

No additional primitive mathematical quantity is required to represent the observables admitted by the framework.

Corollary 2 (Economy of Representation)

Every admitted observable is represented through the same primitive mathematical structure while retaining its own observable mapping.

Accordingly,

$$Q_i = \mathcal{F}_i(S), \qquad (66)$$

or equivalently,

$$Q_i = \mathcal{F}_i(dR_t U_O). \qquad (67)$$

The observable mappings may differ in analytical form, but the primitive mathematical structure upon which they act remains unchanged.

Corollary 3 (Closure of the Mathematical Architecture)

Since every admitted observable possesses a representation

$$Q_i = \mathcal{F}_i(S), \qquad (68)$$

and no additional primitive mathematical quantity is required, every admitted observable belongs to the common mathematical architecture defined by:

1. the mathematical universe;
2. observer universes;
3. admissible physical states;
4. observable mappings; and
5. the representation space.

Accordingly, the mathematical architecture is closed with respect to the observables admitted by the framework.

Remark

The Completeness Theorem establishes the completeness of the mathematical architecture adopted by the TDC framework only within its stated domain of applicability. It does not assert that alternative physical theories cannot employ different primitive mathematical quantities, nor does it claim that future theoretical or observational developments cannot motivate extensions of the present framework.

Rather, the theorem establishes that, under the definitions and axioms adopted in the present work, the existing primitive mathematical structure is sufficient to represent every observable admitted by the theory.

The following section examines the scope and limitations of this conclusion and distinguishes mathematical completeness from broader questions concerning empirical adequacy and physical completeness.

9. Scope and Limitations of the Completeness Theorem

The Completeness Theorem established in the preceding section concerns the formal mathematical architecture of the Time Dilation Cosmology framework. Its conclusions are therefore restricted to the definitions, axioms, state space, and domain of applicability introduced in the present paper. The theorem should not be interpreted as a statement concerning the completeness of physical science in general.

First, the theorem establishes only that no additional primitive mathematical quantity is required to represent the observables admitted by the TDC framework. It does not establish that no additional derived quantities, auxiliary variables, approximation methods, or computational devices may be introduced for particular applications.

Second, the theorem does not establish that the TDC framework is the only mathematically consistent description of physical phenomena. Other theoretical frameworks may adopt different primitive quantities and remain internally consistent within their own assumptions. The Completeness Theorem applies only to the formal system defined in the present work.

Third, the theorem does not establish the empirical validity of the observable mappings developed in the earlier TDC publications. Whether those mappings accurately describe physical phenomena remains a question of theoretical development, experimental measurement, and observational verification rather than mathematical deduction.

Likewise, the theorem does not establish that future developments cannot extend or generalize the present framework. Additional structures may become useful if the domain of applicability is enlarged beyond the observables, states, and mappings presently admitted by the theory.

The theorem therefore establishes a conditional mathematical result. Under the definitions and axioms adopted in Sections 3 and 4, the admissible state

$$S = (dR_t U_O) \qquad (69)$$

is sufficient to represent every observable admitted by the framework.

The significance of the Completeness Theorem lies in demonstrating the internal closure of the mathematical architecture rather than asserting the universal completeness of physical theory. Within the stated assumptions, every admitted observable is represented through the common mathematical structure established in the preceding sections, and no additional primitive mathematical quantity is required.

Accordingly, the Completeness Theorem should be understood as a statement concerning the deductive organization of the TDC framework rather than as an assertion regarding the ultimate completeness of nature itself.

Remark

The distinction between mathematical completeness and physical completeness is essential to the logical structure of the present work. The theorem establishes only that the mathematical architecture adopted by the TDC framework is complete with respect to the observables admitted by its own definitions and axioms. Questions concerning the ultimate physical adequacy of the framework remain matters for continued theoretical investigation and empirical testing.

Transition to the Conclusion

The Representation and Completeness Theorems together establish the formal mathematical architecture of the Time Dilation Cosmology framework. Having defined the primitive mathematical objects, stated the foundational assumptions, and demonstrated the logical organization of the admitted observable mappings, the principal objective of the present work has been achieved. The concluding section summarizes the mathematical significance of these results and their role in the continuing development of the TDC framework.

10. Conclusion

The objective of the present paper has been to develop a formal mathematical architecture for the Time Dilation Cosmology (TDC) framework. Unlike the earlier papers in the TDC publication series, which developed the physical relationships of the theory, the present work has focused on the logical organization of those relationships within a unified deductive mathematical structure.

The paper began by defining the fundamental mathematical objects employed by the framework, including the mathematical universe, observer universes, the observer-conditioned differential time-rate field, admissible physical states, observable mappings, and the representation space. These definitions established the mathematical vocabulary upon which the remainder of the framework is constructed.

The foundational assumptions of the framework were then stated explicitly as axioms. These axioms identify the primitive mathematical structure adopted by the theory, define the admissible physical state, establish the existence and consistency of observable mappings, and specify the domain of applicability within which the subsequent theorems are valid.

From these definitions and axioms, a sequence of preliminary lemmas was derived to establish the elementary structural properties required for the principal results of the paper.

The **Representation Theorem** demonstrated that every observable admitted by the TDC framework possesses a well-defined mathematical representation through the admissible physical state

$$S = (dR_t U_O). \qquad (70)$$

The **Completeness Theorem** further established that, within the assumptions and domain defined by the framework, no additional primitive mathematical quantity is required to represent the observables admitted by the theory.

Together, these theorems organize the previously developed TDC relationships within a single deductive mathematical architecture while maintaining explicit distinctions between mathematical definitions, foundational assumptions, derived theorems, and physical interpretation. The paper therefore provides a formal mathematical framework within which the observable mappings developed in the earlier TDC publications may be understood, analyzed, and extended.

The conclusions presented here are intentionally limited to the mathematical architecture of the TDC framework. They do not, by themselves, establish the empirical validity or universal applicability of the theory. Those questions remain matters for continued theoretical investigation, observational testing, and experimental evaluation.

Accordingly, the principal contribution of the present work is the formalization of the mathematical structure underlying the Time Dilation Cosmology framework. By presenting its definitions, assumptions, mappings, and principal theorems within a unified deductive system, the paper provides a foundation upon which future mathematical developments and physical investigations of the TDC framework may be constructed.

The mathematical architecture developed here is intended to serve as the formal foundation for subsequent developments of the Time Dilation Cosmology framework, including future mathematical generalizations and continued observational investigation.

Appendix A. Representative Observable Mappings Previously Established in the Time Dilation Cosmology Framework

The Representation Theorem established in Section 6 demonstrates that every observable admitted by the Time Dilation Cosmology (TDC) framework possesses a mathematical representation through an admissible observable mapping. The Completeness Theorem established in Section 8 further demonstrates that no additional primitive mathematical quantity is required to represent the observables admitted by the framework.

This appendix is informative rather than deductive. The mappings summarized below are representative examples developed in the earlier TDC publications. They are not derived in the present paper and are not used in the proofs of the Representation or Completeness Theorems.

Each mapping is an application of the general representation

$$Q_i = \mathcal{F}_i(S), \qquad (A1)$$

where the admissible physical state is

$$S = (dR_t U_O). \qquad (A2)$$

The equation numbers below apply only within this appendix.

A.1 Differential Time-Rate Field

The differential time-rate relation is

$$dR_t = \frac{v^2}{c^2}. \qquad \text{(A3)}$$

Physical interpretation. The differential time-rate field is adopted as the primitive scalar quantity of the TDC framework. The velocity associated with a field value is treated as a derived observable.

Principal source: References [1-3].

A.2 Velocity

Solving Equation (A1) for velocity gives

$$v = c\sqrt{dR_t}. \qquad \text{(A4)}$$

Physical interpretation. Velocity is represented as a consequence of the local differential time-rate field rather than as an independent primitive quantity.

Principal source: References [1-3].

A.3 Proper-Time Rate

With the normalized reference time rate equal to unity, the local proper-time rate is represented by

$$R_t = 1 - dR_t. \qquad \text{(A5)}$$

Physical interpretation. The local clock rate is expressed directly in terms of the differential time-rate field.

Principal source: References [1-3].

A.4 Gravitational Acceleration

The gravitational acceleration mapping is

$$a = c^2 \nabla dR_t. \qquad \text{(A6)}$$

Physical interpretation. Gravitational acceleration is represented as the spatial gradient of the differential time-rate field.

Principal source: Reference [4].

A.5 Gravitational Constant Representation

The gravitational relation developed in the TDC framework is

$$G = \frac{rc^2 dR_t}{M}. \qquad \text{(A7)}$$

Physical interpretation. Newton's gravitational constant is represented through the mass-associated differential time-rate field, the radial distance, and the source mass.

Principal source: Reference [4].

A.6 Hamiltonian Representation

The reduced Hamiltonian mapping is

$$H = \frac{1}{2} mc^2 dR_t. \qquad \text{(A8)}$$

Physical interpretation. The Hamiltonian is represented as a function of the primitive differential time-rate field within the reduced TDC formalism.

Principal source: Reference [9].

A.7 Relativistic Energy

The total-energy representation is

$$E = mc^2 \sqrt{1 + dR_t}. \qquad \text{(A9)}$$

Physical interpretation. Total energy is represented as a function of the differential time-rate field.

Principal source: Reference [9].

A.8 Quantum Phase

The accumulated quantum phase is represented by

$$\Delta\theta = \frac{1}{\hbar}\int_{t_1}^{t_2} H\,(t)\,dt. \qquad \text{(A10)}$$

Substitution of the reduced TDC Hamiltonian gives

$$\Delta\theta = \frac{mc^2}{2\hbar}\int_{t_1}^{t_2} d\,R_t(t)\,dt. \qquad \text{(A11)}$$

Physical interpretation. Quantum phase evolution is represented through the Hamiltonian mapping of the differential time-rate field.

Principal source: Reference [9].

A.9 Cosmological Scale Factor

Within the TDC cosmological interpretation, the FLRW scale factor is represented by

$$a(t) = R_t(t). \qquad \text{(A12)}$$

Physical interpretation. The scale factor is interpreted as the cosmological time-rate field rather than as an independently primitive spatial-expansion quantity.

Principal source: Reference [9].

A.10 Cosmological Redshift

The corresponding cosmological redshift relation is

$$1 + z = \frac{R_t(\text{obs})}{R_t(\text{emit})}. \qquad \text{(A13)}$$

Physical interpretation. Cosmological redshift is represented as the ratio of the observer and emitter time rates.

Principal source: Reference [1,9].

A.11 Einstein Field Representation

The Einstein field equations retain their standard mathematical form,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \qquad \text{(A14)}$$

Within TDC, the quantities entering this equation are interpreted through the differential time-rate formulation developed in Reference [9].

Physical interpretation. The standard tensor equation is retained while its physical quantities are organized within the differential time-rate representation.

Principal source: Reference [9].

A.12 Observer-Conditioned Physical State

The admissible physical state used in the present paper is

$$S = (dR_t U_O) \qquad \text{(A15)}$$

Physical interpretation. An observable representation depends upon both the primitive differential time-rate field and the observer universe containing the applicable boundary, initial, and observational conditions.

Principal source: Present paper.

A.13 General Observable Representation

Every admitted observable is organized through the general mapping

$$Q_i = \mathcal{F}_i(dR_t U_O). \qquad \text{(A16)}$$

Physical interpretation. Individual observables may have different analytical forms while sharing the same primitive mathematical state.

Principal source: Present paper.

Appendix Summary

The mappings summarized above illustrate the application of the Representation and Completeness Theorems to previously developed TDC relationships. They include representative quantities associated with relativistic velocity, proper-time rate, gravitation, energy, Hamiltonian mechanics, quantum phase evolution, FLRW cosmology, cosmological redshift, and the Einstein field equations.

These examples do not constitute proofs of the individual physical relationships. Their derivations remain in the cited TDC publications. Their purpose here is to demonstrate how distinct observable mappings may be organized within the common formal structure

$$Q_i = \mathcal{F}_i(dR_t U_O). \qquad (A17)$$

The observable mappings retain their individual analytical forms while belonging to the common representation space established in the present paper.

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