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The Cyclic Universe Driven by Loop Quantum Cosmology

Yongge Ma, Ph.D.
Department of Physics, Beijing Normal University, Beijing 100875, China

The loop quantum cosmological effects which might lead to a cyclic universe are reviewed. We concentrate on isotropic loop quantum cosmology coupled with a massless scalar field. By semiclassical analysis, the effective Hamiltonian constraints for different proposed Hamiltonian operators are obtained, which incorporate also the next to leading order quantum corrections. It turns out that the classical big bang singularity will get replaced by a quantum bounce in all scenarios. Moreover, if the semiclassicality of the model is maintained in the large scale limit, there are great possibilities for $k = 0$ Friedmann expanding universe to undergo a recollapse in the future due to the quantum gravity effect. Thus the quantum bounce and recollapse may contribute a cyclic universe. The above results of canonical effective Hamiltonian constraints can also be justified by the coherent state path-integral approach of loop quantum cosmology.

Keywords: Big Bang, Cyclic Universe, Big Crunch, Collapsing Universe

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The loop quantum cosmological effects which might lead to a cyclic universe are reviewed. We concentrate on isotropic loop quantum cosmology coupled with a massless scalar field. By semiclassical analysis, the effective Hamiltonian constraints for different proposed Hamiltonian operators are obtained, which incorporate also the next to leading order quantum corrections. It turns out that the classical big bang singularity will get replaced by a quantum bounce in all scenarios. Moreover, if the semiclassicality of the model is maintained in the large scale limit, there are great possibilities for $k = 0$ Friedmann expanding universe to undergo a recollapse in the future due to the quantum gravity effect. Thus the quantum bounce and recollapse may contribute a cyclic universe. The above results of canonical effective Hamiltonian constraints can also be justified by the coherent state path-integral approach of loop quantum cosmology.

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As a background independent quantum theory of gravity, loop quantum gravity (LQG) has been rather active in recent twenty-five years (Ashtekar et al. 2004; Han et al. 2007; Rovelli, 2004; Thiemann, 2007). The expectation that the singularities predicted by classical general relativity (GR) would be resolved by some quantum gravity theory has been confirmed by the recent study of certain isotropic models in loop quantum cosmology (LQC) (Bojowald, 2001; Ashtekar et al. 2006a,b), which is a simplified symmetry-reduced model of LQG (Bojowald, 2005; Ashtekar, 2009). The basic purpose of LQG is to merge the conceptual insight of GR into quantum mechanics. To achieve this purpose, one only makes use of the general tools of a quantum theory. The Hilbert space and operators are obtained from classical GR following certain quantization strategy. In contrast to the initial Wheeler-DeWitt canonical quantization of GR (Wheeler, 1962; Dewitt, 1967), the classical algebra that one wants to represent on the Hilbert space of LQG is based on the holonomies of the gravitational connection. Physically, holonomies are natural variables representing Faraday's 'lines of force', that do not refer to what happens at a point, but rather refer to the relation between different points connected by a line. Mathematically, the quantum configuration space of LQG can be constructed by the concept of holonomy, since its definition does not depend on an extra background. It turns out that the kinematical framework of LQG can be established with mathematical rigour.

In the models of LQC, the idea that one should view holonomies rather than connections as basic variables for the quantization of gravity is successfully carried on (Bojowald, 2000; Ashtekar et al. 2003a). In a LQC scenario for a universe filled with a massless scalar field, the classical singularity gets replaced by a quantum bounce (Ashtekar et al. 2006a,b). Various features of the bounce have been revealed through different considerations (Bojowald, 2007a,b,2008; Corichi et al. 2008). While the model shows that the quantum effect played the key role in Planck scale to cure the big bang singularity as one expected, the question whether quantum gravity effect can also be manifested in large scale cosmology is being studied (Ding et al. 2009; Yang et al. 2009a,b). This question is crucial since, besides overcoming the difficulties of a classical theory, to predict phenomena which are dramatically different from those of the classical theory is also a hallmark to identify a quantum theory.

We are going to review the loop quantum cosmological effects which might lead to a cyclic universe in spatially flat ($k = 0$) FRW model. We consider the so-called improved dynamics framework of LQC (Ashtekar et al. 2006b). In the kinematical setting, one has to introduce an elementary cell $\mathcal{V}$ and restricts all integrations to this cell. Fix a fiducial flat metric ${}^o q_{ab}$ and denote by V_o the volume of $\mathcal{V}$ in this geometry. The gravitational phase space variables —the connections and the density-weighted triads — can be expressed as $A_a^i = c\, V_o^{-(1/3)}\ {}^o\omega_a^i$ and $E_i^a = p\, V_o^{-(2/3)} \sqrt{{}^o q}\ {}^o e_i^a$, where $({}^o\omega_a^i, {}^o e_i^a)$ are a set of orthonormal co-triads and triads compatible with ${}^o q_{ab}$ and adapted to $\mathcal{V}$. p is related to the scale factor a via $|p| = V_o^{2/3} a^2$. The fundamental Poisson bracket is given by: $\{c, p\} = 8\pi G\gamma/3$, where G is the Newton's constant and γ the Barbero-Immirzi parameter. The gravitational part of the Hamiltonian constraint reads $H_{\rm grav} = -6c^2\sqrt{|p|}/\gamma^2$. It is convenient to introduce new conjugate variables by a canonical transformation:

$$b := \frac{\sqrt{\Delta}}{2}\frac{c}{\sqrt{|p|}}, \qquad \nu := \frac{4}{3\sqrt{\Delta}}\,{\rm sgn}(p)|p|^{\frac{3}{2}},$$

where $\Delta \equiv (4\sqrt{3}\pi\gamma)\,\ell_{\rm P}^2$ is the 'area gap' from full LQG (Ashtekar, 2009) and $\ell_{\rm P}^2 = G\hbar$.

In the kinematical Hilbert space $\mathcal{H}_{\rm kin}^{\rm grav}$ of the quantum theory, eigenstates of $\hat{\nu}$, which are labelled by real numbers v, constitute an orthonormal basis as: $\langle v_1|v_2\rangle = \delta_{v_1,v_2}$. The fundamental operators act on $|v\rangle$ as: $\hat{\nu}\,|v\rangle = (8\pi\gamma\ell_{\rm P}^2/3)v|v\rangle$ and $\widehat{e^{ib}}\,|v\rangle = |v+1\rangle$. In the improved

*Electronic address: mayg@bnu.edu.cn

LQC treatments, there are different proposed versions of the gravitational part of the Hamiltonian operators (Ashtekar et al. 2006b; Yang et al. 2009a,b). The one which inherits more features from the full LQG contains the Euclidean term $\hat{H}^E(1)$ and the Lorentzian term $\hat{T}_F(1)$ (Yang et al. 2009a,b). The action of this gravitational Hamiltonian constraint operator on $|v\rangle$ is given by (Yang et al. 2009b)

$$\begin{aligned}\hat{H}^F_{\rm grav}|v\rangle =&\hat{H}^E_{\rm grav}(1)|v\rangle - 2(1+\gamma^2)\hat{T}_F(1)|v\rangle \\ =&F'_+(v)|v+8\rangle + f'_+(v)|v+4\rangle \\ &+ \left[F'_o(v)+f'_o(v)\right]|v\rangle \\ &+ f'_-(v)|v-4\rangle + F'_-(v)|v-8\rangle, \end{aligned} \quad (1)$$

where $F'_*(v)$ and $f'_*(v)$ are certain functions of v. As in Ashtekar et al. (2006a,b), to identify a dynamical matter field as an internal clock, we take a massless scalar field ϕ with Hamiltonian $H_\phi = |p|^{-\frac{3}{2}}p^2_\phi/2$, where p_ϕ denotes the momentum of ϕ. While we choose the standard Schrödinger representation for ϕ, the operator $\widehat{1/|p|^{3/2}}$ is diagonal in the v representation. Then we can express the matter part of the quantum Hamiltonian constraint as $\hat{H}_\phi = \frac{1}{2}\widehat{|p|^{-\frac{3}{2}}}\,\widehat{p^2_\phi}$ and the total constraint as $\hat{H}^F = \frac{1}{16\pi G}\hat{H}^F_{\rm grav} + \hat{H}_\phi$.

To do the canonical semiclassical analysis, we need to calculate the expectation value of the Hamiltonian constraint operator with respect to suitable semiclassical states. A semiclassical state $(\Psi_{(b_o,\nu_o)}|$ peaked at a point (b_o,ν_o) of the gravitational classical phase space reads:

$$(\Psi_{(b_o,\,\nu_o)}| = \sum_{v\in\mathbb{R}} e^{-\frac{(v-v_o)^2}{2d^2}}\, e^{ib_o(v-v_o)}(v|, \quad (2)$$

where $d = 1/\epsilon$ is the characteristic 'width' of the coherent state, and v_o is related to ν_o through $\nu_o = (8\pi\gamma\ell^2_{\rm P}/3)v_o$. For practical calculations, we use the shadow of the semiclassical state $(\Psi_{(b_o,\nu_o)}|$ on the regular lattice with spacing 1 (Ashtekar et al. 2003b). The semiclassical state of matter part is given by the standard coherent state

$$(\Psi_{(\phi_o,p_\phi)}| = \int {\rm d}\phi\, {\rm e}^{-\frac{(\phi-\phi_o)^2}{2\sigma^2}}\, {\rm e}^{\frac{\rm i}{\hbar}{\rm p}_\phi(\phi-\phi_o)}(\phi|, \quad (3)$$

where σ is the width of the Gaussian. Thus the whole semiclassical state reads $(\Psi_{(b_o,\,\nu_o)}|\bigotimes(\Psi_{(\phi_o,p_\phi)}|$. It turns out that, by using the above semiclassical state, we can obtain an effective Hamiltonian with the relevant quantum corrections of order $\epsilon^2, 1/v^2\epsilon^2, \hbar^2/\sigma^2p^2_\phi$ as (Yang et al. 2009b):

$$\begin{aligned}\mathcal{H}^F_{\rm eff} =& -\frac{3^2\sqrt{6}}{2^3}\frac{\hbar^{1/2}}{\gamma^{3/2}\,\kappa^{1/2}}L\,|v| \\ &\times\Big[\sin^2(2b)\big[1-(1+\gamma^2)\sin^2(2b)\big]+2\epsilon^2\Big] \\ &+\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\frac{|v|}{L}\rho\left(1+\frac{1}{2|v|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p^2_\phi}\right), \end{aligned} \quad (4)$$

where $\kappa = 8\pi G$ and $L = \frac{4}{3}\sqrt{\frac{\pi\gamma\ell^2_p}{3\Delta}}$. It is easy to see that the classical constraint is reproduced up to small quantum corrections, and therefore the Hamiltonian operator $\hat{H}_F$ has correct classical limit. We can further obtain the Hamiltonian evolution equation of v by taking its Poisson bracket with $\mathcal{H}^F_{\rm eff}$ as

$$\dot{v}_F = 3|v|\sqrt{\frac{\kappa}{3}\rho_c}\,\sin(2b)\cos(2b)\Big[1-2(1+\gamma^2)\sin^2(2b)\Big], \quad (5)$$

where $\rho_c = 3/(\kappa\gamma^2\Delta)$. The vanishing of the effective Hamiltonian constraint (4) gives

$$\begin{aligned}&\sin^2(2b)\big[1-(1+\gamma^2)\sin^2(2b)\big] \\ &\quad = \frac{\rho}{\rho_c}\left(1+\frac{1}{2|v|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p^2_\phi}\right) - 2\epsilon^2. \end{aligned} \quad (6)$$

For the semiclassical regime, $b \ll 1$ and $\rho \ll \rho_c$, from Eq.(6) we have

$$\sin^2(2b) = \frac{1-\sqrt{1-\chi_F}}{2(1+\gamma^2)}, \quad (7)$$

where

$$\chi_F = 4(1+\gamma^2)\left[\frac{\rho}{\rho_c}\left(1+\frac{1}{2|v|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p^2_\phi}\right)-2\epsilon^2\right]. \quad (8)$$

The modified Friedmann equation can then be derived as

$$\begin{aligned}H^2_F =& \left(\frac{\dot{v}_F}{3v}\right)^2 \\ =& \frac{\kappa}{3}\frac{\rho_c}{4(1+\gamma^2)^2}\left(1-\sqrt{1-\chi_F}\right)\left(1+2\gamma^2+\sqrt{1-\chi_F}\right) \\ &\times(1-\chi_F)\,. \end{aligned} \quad (9)$$

It is easy to see that if one neglects the small quantum corrections in the classical region, $\chi_F \ll 1$ for $\rho \ll \rho_c$, one gets

$$H^2_F \approx \frac{\kappa}{3}\frac{\rho_c}{4(1+\gamma^2)^2}\frac{1}{2}\chi_F 2(1+\gamma^2) \approx \frac{\kappa}{3}\rho, \quad (10)$$

which reduces to the standard Friedmann equation. However, quantum geometry effects lead to a modification of the Friedmann equation especially at the scales when ρ becomes comparable to ρ_c. Remarkable changes to the classical theory happen when the Hubble parameter in Eq. (9) vanishes by

$$1-\chi_F = 0. \quad (11)$$

If we consider only the leading order contribution in Eq. (8), this can happen when

$$\rho = \rho_c/4(1+\gamma^2). \quad (12)$$

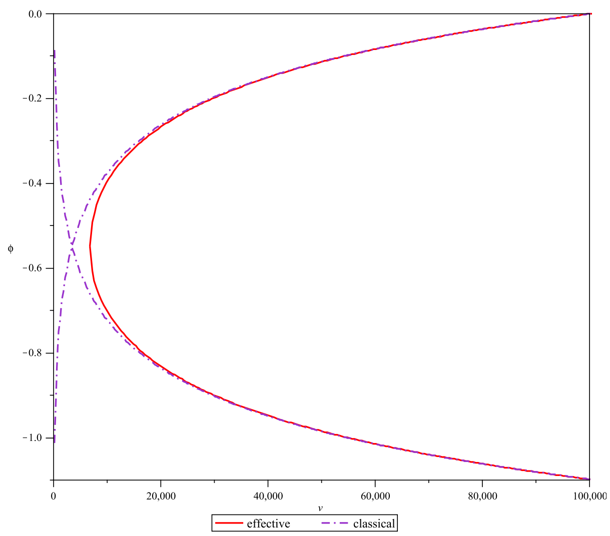

FIG. 1: The effective dynamics represented by the observable $v|_\phi$ are compared to classical trajectories. In this simulation, the parameters were: $G = \hbar = 1\,,\, p_\phi = 10\,000$, $\epsilon = 0.001\,,\sigma = 0.01$ with initial data $v_o = 100\,000$.

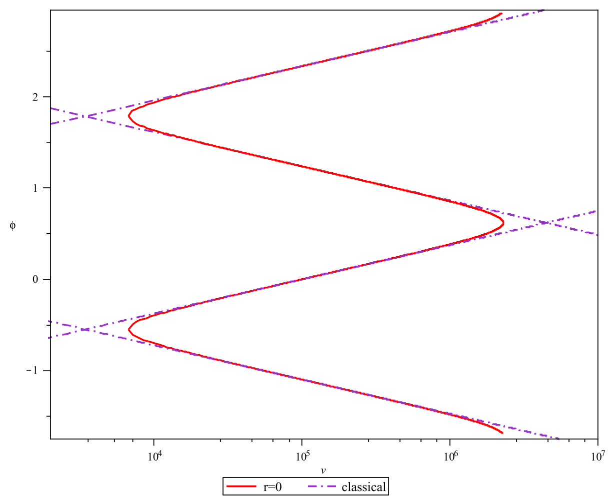

FIG. 2: The cyclic model is compared with expanding and contracting classical trajectories. In this simulation, the parameters were: $G = \hbar = 1\,,\, p_\phi = 10\,000\,, \epsilon = 0.001, \sigma = 0.01$ with initial data $v_o = 100\,000$.

Thus, when energy density of the scalar field reaches to the leading order critical energy density $\rho_c^F = \rho_c/4(1+\gamma^2)$, the universe bounces from the contracting branch to the expanding branch. The quantum bounce implied by (9) is shown in Fig. 1 .

It is easy to see that Eq.(9) may also reduce to the leading order effective Friedmann equation $H^2 = (8\pi G/3)\rho(1-\rho/\rho_c)$, if the terms of order $1/(v^2\epsilon^2)$, ϵ^2 and $\hbar^2/\sigma^2 p_\phi^2$ are neglected. However, as we will see, the minus sign in front of the ϵ^2 term in Eq.(8) may lead to a qualitatively different scenario from the leading order effective theory. Thus these subleading terms cannot be neglected at will, while those neglected higher order corrections cannot lead to qualitatively different effect. For an expanding universe, it is easy to see from Eq. (9) that the Hubble parameter may also vanish by the vanishing of χ_F, which would lead to a collapse point in our scenario. The quantum fluctuations or the Gaussian spread ϵ plays a key role here. Thus its concrete form becomes rather relevant. One usually sets the innocent condition that the relative spreads of the basic conjugate variables are small for semiclassical states, i.e., $\frac{\Delta v}{v} \sim \frac{1}{\sqrt{2}\epsilon v} \ll 1$ and $\frac{\Delta b}{b} \sim \frac{\epsilon}{\sqrt{2}b} \ll 1$. A simple setting could be $\epsilon = \lambda(r)v^{-r(\phi)}$, where $0 \le r(\phi) \le 1$ and the parameter $\lambda(r)$ has to be suitably chosen for different value of r. We now illustrate the extreme case where $r = 0$. Besides the quantum bounce when the matter density ρ increases to the Planck scale, the universe would also undergo a recollapse when ρ decreases to $\rho_{\rm coll}^F \approx 8(1+\gamma^2)\epsilon^2\rho_c^F$. Therefore the quantum fluctuations lead to a cyclic universe in this case as illustrated in Fig. 2.

Since the significant departure occurs only at the large scale limit, the asymptotic behavior of $r(\phi)$ is crucial. It is easy to see from Eq.(8) that an expanding universe would undergo the recollapse and become cyclic provided $0 \le r < 1$ asymptotically. Suppose that the semiclassicality of our coherent state is maintained in the large scale limit. This implies that the quantum fluctuation $1/\epsilon$ of v cannot increase as v unboundedly as v approaches infinity. This is another way of saying that the quantum fluctuation ϵ of $\hat{b}$ cannot approach zero as b, since otherwise the coherent state would approach an eigenstate of $\hat{b}$ and thus lose its coherence. In fact, the innocent condition $\frac{\Delta b}{b} \ll 1$ is not valid when b approaches zero. This fact is obvious if one recalls the standard coherent states of a harmonic oscillator, where the fluctuation Δx is a constant and hence $\frac{\Delta x}{x} \ll 1$ is not valid when x approaches zero. Therefore, the assumption that the semiclassicality of the model is maintained in the large scale limit indicates a cyclic universe driven by the quantum fluctuations. This inference is in all probability as viewed from the parameter space of $r(\phi)$. This is an amazing possibility that quantum gravity manifests herself in the large scale cosmology, which has never been realized before.

As in the ordinary quantization procedure, there are quantization ambiguities in constructing the Hamiltonian constraint operator for LQC. Hence it is crucial to check whether the key features of LQC in this model, that the big bang singularity is replaced by a quantum bounce and there are great possibilities for an expanding universe to recollapse, are robust against the quantization ambiguities. To this aim, an alternative gravitational Hamiltonian constraint operator $\hat{H}^S_{\rm grav}$ for LQC was also proposed in Yang et al. (2009b) by a different regularization procedure. A similar semiclassical analysis of $\hat{H}^S_{\rm grav}$

led to a modified Friedmann equation as

$$H_S^2 = \frac{\kappa}{3}\frac{\rho_c}{\gamma^4}(-1+\sqrt{1+\chi_S}\,)(1+2\gamma^2-\sqrt{1+\chi_S})(1+\chi_S), \tag{13}$$

where

$$\chi_S = \gamma^2\left[\frac{\rho}{\rho_c}\left(1+\frac{1}{2|v|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2 p_\phi^2}\right)-\frac{1}{2}\epsilon^2\right]. \tag{14}$$

The Hubble parameter in Eq. (13) can also vanish when $1+2\gamma^2-\sqrt{1+\chi_S}=0$ or $\chi_S=0$. While the former corresponds to the bounce point, the latter corresponds to the recollapse point. Thus the quantum dynamics given by $\hat{H}_S$ has qualitatively similar feature of that given by $\hat{H}_F$. Also, the original improved gravitational Hamiltonian constraint operator proposed in Ashtekar et al. (2006b) is essentially the Euclidean term $\hat{H}^E_{\text{grav}}(1)$ in Eq.(1). Its semiclassical analysis led to the following modified Friedmann equation (Ding et al 2009):

$$H^2 = \frac{8\pi G}{3}\rho\Big[1-\frac{\rho}{\rho_c}(1+\frac{1}{v^2\epsilon^2})+\frac{1}{2v^2\epsilon^2}-2\epsilon^2\frac{\rho_c}{\rho}\Big]. \tag{15}$$

It is easy to see that Eq.(15) also gives the quantum dynamics qualitatively similar to the above two cases. Therefore, the key features of LQC in this model, that may drive a cyclic universe, are robust against the quantization ambiguities. Note that, to confirm the above results of canonical semiclassical analysis, we are developing a coherent state path-integral approach for LQC (Qin and Ma, 2011). It turns out that the same effective Hamiltonian constraint can also be derived by the coherent state path-integral under certain condition.

We summarize with a few remarks: (i) For the scenarios of the cyclic universe, the expectation value, infinitesimal Ehrenfest and small fluctuation properties of the shadow coherent state with respect to $\hat{b}$ and $\hat{v}$ are all maintained at both the quantum bounce and recollapse points (Ding et al. 2009). Thus the universe could be semiclassical all the way and present its consistency. (ii) People used to think that quantum gravity could only take effect at small (Planck) scale. While the quantum bounce looks quite natural, one may suspect how quantum effect can change the large scale behavior of the universe. The intuitive picture that we gained from this model is the following. As the universe expands unboundedly, the matter density would become so tiny that its effect could be comparable to that of quantum fluctuations of the spacetime geometry. Then the Hamiltonian constraint may force the universe to contract back. (iii) Caveats may arise from our effective approach. Our confidence arise from the following facts. First, the Planck scale quantum bounce predicted by the effective Friedmann equation (15) has been confirmed by the numerical simulation in the full quantum difference-differential system of this model (Ashtekar et al. 2006b), while the effective Hamiltonian is more accurate for large volumes and late times. Secondly, the same effective Hamiltonian constraint can also be derived by the coherent state path-integral approach under certain condition. Nevertheless, the condition that the semiclassicality is maintained in the large scale limit has not been confirmed for any of above quantum dynamics. Hence further numerical and analytic investigations to the properties of dynamical semiclassical states in the model are desirable. It should be noted that in some simplified completely solvable models of LQC (see Bojowald (2007a,b) and Ashtekar et al. (2008)), the dynamical coherent states could be obtained, where $r(\phi)$ approaches 1 in the large scale limit. While those treatments lead to the quantum dynamics different from ours, they raise caveats to the conjectured recollapse.

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