Journal of Cosmology, Vol. 26, No. 24, pp 15250 - 15268

New Tolman-Oppenheimer-Volkoff Formula in Einstein Real Gravity

YVAN LEBLANC

Email: *yleblanc@efieldtheory.com*
Affiliation: *eFieldTheory.COM*

PACS: 4.60.+n, 11.17.+y, 97.60.Lf

May 30, 2022

Abstract

Einstein Real Gravity (RG) is Einstein General Relativity (GR) with an exponential parametrization of the metric tensor. The tensor potential field in the argument of the exponential is interpreted as the physical spin-2 gravitational field. This allows the linear separation of the spacetime curving real gravitational tensor force from the other non-tensor non spacetime curving inertial forces in the Christoffel symbols. The use of cartesian coordinates filters out these non-tensor forces and leads to a description of GR as the pure RG affine tensor theory. A modified expression for the hydrostatic Tolman-Oppenheimer-Volkoff formula for stellar equilibrium is derived and applied to the ultra-relativistic spherical perfect fluid case. An exact analytical solution is found.

1 Einstein Real Gravity Theory

Einstein General Relativity (GR) describes the laws of nature as seen from general reference frames, whether inertial or not. The equivalence principle assures us that the general motion of particles or material bodies under (loosely called) gravitational forces are equivalently described by observers in non inertial reference frames. But these loosely called gravitational forces covered by the equivalence principle are known not to be restricted to real (pure or actual) gravity. The equivalence principle in fact unifies the real gravitational force with other forces known as inertial forces such as the centrifugal and Coriolis forces of classical mechanics. All of these forces are traditionnally called gravitational forces by general relativists.

The real gravitational force is very different from the other inertial forces. Unlike the latters, the real gravitational force vanishes at infinity [1]. Furthermore, unlike the others, it cannot be removed globally by a change of reference frame. The other inertial forces do disappear when going to an inertial reference frame (cartesian or *natural* coordinates [1, 2]). The reason why the real gravitational force connot be made to disappear is because it *curves spacetime* [1, 2, 3, 4]. The other forces do not, although they remain unified with the real gravitational force through general relativity and the equivalence principle.

One realizes that the equivalence principle is too wide a principle, englobing too many physical forces. The description of real gravity is then cluttered by these unwanted forces which complicate unnecessarily the dynamics. It thus seems reasonable to split general relativity into two parts [3, 4, 5, 6, 7], one covering the flat spacetime inertial forces which should be incorporated into a generalization of special relativity, and another one dealing exclusively with curved spacetime phenomena, *i.e.* the real gravitational force which we call Einstein Real Gravity (RG).

Such a linear separation of forces can only be accomplished with the aid of the Yilmaz exponential parametrization of the metric tensor [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], written in terms of so-called *tensor potentials* $\phi_i^{\ j}$ as follows,

$$g_{ij} \ = \ (e^{2(\phi\hat{I}-2\hat{\phi})})_i^{\ k}\,\eta_{kj} \ = \ \psi_i^{\ k}[\hat{\Sigma}]\,\eta_{kj}\ , \tag{1.1}$$

$$\psi_i^{\ k}[\hat{\Sigma}] \ \equiv \ (e^{-4\hat{\Sigma}})_i^{\ k}\ , \tag{1.2}$$

$$(\hat{\Sigma})_i^{\ j} \ = \ \bar{\phi}_i^{\ j} \ \equiv \ \phi_i^{\ j} - \frac{1}{2}\delta_i^{\ j}\phi\ , \tag{1.3}$$

in which $\eta_{ij} = (-1, 1, 1, 1)$ is the flat spacetime metric in cartesian coordinates (which defines a galilean reference frame [1]) and where $(\hat{\phi})_i^j = \phi_i^j$ and $(\hat{I})_i^j = \delta_i^j$ with trace $\phi \equiv \phi_i^{\ i} = -\bar{\phi}$. Note that the metric determinant $\sqrt{-g}$ is given by $\sqrt{-g} = e^{2\phi} = e^{-2\bar{\phi}}$.

As discussed in previous works [3, 4, 5, 6, 7], the Yilmaz tensor potentials ϕ_i^j are related *linearly* to Newton's gravitational potential, unlike the traditional metric representation of GR. To lowest order in Newton's constant, the traditional metric representation is only the first order term of the exponential parametrization representation and so constitutes only a weak field description. Note that both representations, to this point, equally agree with all observational and experimental tests of weak field general relativity [13].

Unlike the traditional weak field metric representation, the strong field exponential parametrization displays no event horizon. Therefore black holes do not exist in this parametrization and no singularity ever develop [9, 10, 11, 12, 13, 14, 15, 16]. The universe is everywhere regular, in complete agreement with basic observations.

Let us now take into consideration the existence of the two physically independent and separate gravitational potentials representing respectively the pure or *real gravitational potentials* φ_i^j and the coordinates or *inertial potentials* χ_i^j expressed in the following manner [3, 4, 5, 6, 7],

$$\hat{\Sigma} \;=\; \hat{\Phi} + \hat{\Omega} \quad ; \quad \phi_i^{\,j} \;=\; \varphi_i^{\,j} + \chi_i^{\,j} \;\; , \tag{1.4}$$

leading to the following metric tensor,

$$g_{ij} \;=\; \psi_i^{\,k}[\hat{\Sigma}]\,\eta_{kj} \;=\; \psi_i^{\,k}[\hat{\Phi}+\hat{\Omega}]\,\eta_{kj} \; , \tag{1.5}$$

with the definitions,

$$(\hat{\Phi})_i^{\,j} \;=\; \bar{\varphi}_i^{\,j} \;\equiv\; \varphi_i^{\,j} \;-\; \frac{1}{2}\delta_i^{\,j}\varphi \; , \tag{1.6}$$

$$(\hat{\Omega})_i^{\,j} \;=\; \bar{\chi}_i^{\,j} \;\equiv\; \chi_i^{\,j} \;-\; \frac{1}{2}\delta_i^{\,j}\chi \; . \tag{1.7}$$

For diagonal metrics such as the central symmetric isotropic spacetime [5, 7], the gravitational and inertial potentials are also diagonal and so eq. (1.5) for the tensor potentials $\phi_i^{\,j}$ cleanly factorizes as follows,

$$g_{ij} \;=\; \psi_i^{\,k}[\hat{\Sigma}]\,\eta_{kj} \;=\; \psi_i^{\,k}[\hat{\Phi}+\hat{\Omega}]\,\eta_{kj} \;=\; \psi_i^{\,k}[\hat{\Phi}]\,\eta_{kj}[\hat{\Omega}], \tag{1.8}$$

where we defined,

$$\eta_{kj}[\hat{\Omega}] \;\equiv\; \psi_k^{\,i}[\hat{\Omega}]\,\eta_{ij} \;\; , \tag{1.9}$$

which is a coordinate transformation from inertial (cartesian) to non inertial (spherical, cylindrical, etc.) flat spacetime coordinates.

Yilmaz exponential parametrization of the metric now leads to the following expression for the Christoffel symbols in terms of the tensor potentials $\bar{\phi}_i^{\,j}$,

$$\Gamma^i_{\;kl}[\hat{\Sigma}] \;=\; -4\partial_{(l}\bar{\phi}^{\,i}_{k)} \;+\; 2g^{im}\partial_m\bar{\phi}^{\,j}_{(l}g_{jk)} \; . \tag{1.10}$$

Recalling the separation (1.4)-(1.7) in terms of the real gravitational potentials $\bar{\varphi}_i^j$ and the inertial potentials $\bar{\chi}_i^j$, we get the following linear separation of the Christoffel symbols,

$$\Gamma^i_{\;kl}[\hat{\Sigma}] \;=\; \Delta^i_{\;kl}[\hat{\Phi}] + S^i_{\;kl}[\hat{\Omega}] \; , \tag{1.11}$$

in which we defined the *Gravitational symbols* $\Delta^i_{\;kl}$ and *Inertial symbols* $S^i_{\;kl}$ as follows,

$$\Delta^i_{\;kl}[\hat{\Phi}] \;\equiv\; -4\partial_{(l}\bar{\varphi}^{\,i}_{k)} \;+\; 2g^{im}\partial_m\bar{\varphi}^{\,j}_{(l}g_{jk)} \; , \tag{1.12}$$

and,

$$S^i_{\;kl}[\hat{\Omega}] \;\equiv\; -4\partial_{(l}\bar{\chi}^{\,i}_{k)} \;+\; 2g^{im}\partial_m\bar{\chi}^{\,j}_{(l}g_{jk)} \; . \tag{1.13}$$

For general and inertial (cartesian) coordinates, we have respectively,

$$\Gamma^i_{\;kl} = S^i_{\;kl} + \Delta^i_{\;kl} \; , \tag{1.14}$$

$$\Gamma'^{i}{}_{kl} = S'^{i}{}_{kl} + \Delta'^{i}{}_{kl} \ , \tag{1.15}$$

with the following transformation formula,

$$\Gamma^{i}{}_{kl} = \Gamma'^{s}{}_{rp} \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial x^i}{\partial x'^s} \frac{\partial^2 x'^s}{\partial x^k \partial x^l} \ , \tag{1.16}$$

and so,

$$S^{i}{}_{kl} + \Delta^{i}{}_{kl} = (S'^{s}{}_{rp} + \Delta'^{s}{}_{rp}) \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial x^i}{\partial x'^s} \frac{\partial^2 x'^s}{\partial x^k \partial x^l} \ . \tag{1.17}$$

Now the Inertial symbols are pure coordinate transformations from inertial (cartesian) x'^s to general x^k coordinates,

$$S^{i}{}_{kl} = \frac{\partial x^i}{\partial x'^s} \frac{\partial^2 x'^s}{\partial x^k \partial x^l} \ , \tag{1.18}$$

which leads directly to $S'^{s}{}_{rp} = 0$ in inertial (cartesian) coordinates. This is so because the only transformations between inertial coordinates are linear Lorentz transformations, which is the statement of vanishing inertial (non-tensor) forces in inertial coordinate systems.

Now since $S'^{s}{}_{rp} = 0$, eqs. (1.17)-(1.18) further imply that the Gravitational symbols $\Delta^{i}{}_{kl}$ are themselves pure tensors,

$$\Delta^{i}{}_{kl} = \Delta'^{s}{}_{rp} \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^p}{\partial x^l} \ , \tag{1.19}$$

which means that *the real gravitational force is a pure tensor force* [3, 4, 5, 6, 7].

The fundamental field equation of Einstein Real Gravity

It is well known that the Einstein tensor $G_i^{\ k}$ can be decomposed in terms of the Freud $F_i^{\ k}$ [3, 4, 5, 6, 7, 9, 17] and Einstein-Pauli $E_i^{\ k}$ [1, 2, 3, 4, 5, 6, 7, 9, 18] pseudo-tensors as follows,

$$\sqrt{-g} G_i^{\ k} \ = \ \sqrt{-g} (F_i^{\ k} \ - \ E_i^{\ k}) \ . \tag{1.20}$$

The Einstein-Pauli pseudo-tensor is given as [3, 4, 5, 9],

$$\begin{aligned} \sqrt{-g}\, E_j^{\ k} \quad &= \quad \sqrt{-g} \left(W_j^{\ k} - \frac{1}{2} \delta_j^{\ k} W \right) \\ \sqrt{-g}\, E \quad &\equiv \quad \sqrt{-g}\, E_j^{\ k} \delta_k^{\ j} \ = \ -\sqrt{-g}\, W \ , \end{aligned} \tag{1.21}$$

where,

$$\begin{aligned} \sqrt{-g}\, W_j^{\ k} \quad &\equiv \quad -8\, \mathbf{g}^{ks} \left(\partial_j \phi_l^{\ m} \partial_m \phi_s^{\ l} - \frac{1}{2} \partial_j \phi_l^{\ m} \partial_s \phi_m^{\ l} + \frac{1}{4} \partial_j \phi \partial_s \phi \right) \\ \frac{1}{2} \sqrt{-g}\, W \quad &= \quad -4\, \mathbf{g}^{rs} \left(\partial_l \phi_r^{\ m} \partial_m \phi_s^{\ l} - \frac{1}{2} \partial_r \phi_l^{\ m} \partial_s \phi_m^{\ l} + \frac{1}{4} \partial_r \phi \partial_s \phi \right) \ , \end{aligned} \tag{1.22}$$

and in which we defined the **rescaled metric tensor**,

$$\mathbf{g}^{ij} \equiv \sqrt{-g} g^{ij} \ \ ; \ \ \mathbf{g}_{ij} \equiv \frac{1}{\sqrt{-g}} g_{ij} \ . \tag{1.23}$$

The Freud pseudo-tensor on the other hand can be expressed in the following manner [3, 4, 5, 6, 7, 9, 17, 18],

$$\sqrt{-g}F_j^{\ k} \ \equiv \ \partial_l(\sqrt{-g}B_j^{\ kl}) \ , \tag{1.24}$$

with the antisymmetric super-potential $B_j^{\ kl}$ given by [3, 4, 5, 6, 7, 9, 18],

$$\begin{aligned}\sqrt{-g}B_j^{\ kl} = \sqrt{-g}B_j^{\ [kl]} \ &= \ -\frac{1}{2}\sqrt{-g}\,\big[\,\delta_j^{\ k}(g^{rs}\Gamma^l_{\ rs} - g^{lr}\Gamma^s_{\ rs}) \\ &\quad + \ \delta_j^{\ l}(g^{kr}\Gamma^s_{\ rs} - g^{rs}\Gamma^k_{\ rs}) \\ &\quad + \ (g^{lr}\Gamma^k_{\ jr} - g^{kr}\Gamma^l_{\ jr})\big] \ ,\end{aligned} \tag{1.25}$$

with the property $\partial_k\partial_l(\sqrt{-g}B_j^{\ kl}) = 0$. The antisymmetry of the super-potential in turn leads directly to the so-called Freud identity [3, 4, 5, 6, 7, 9],

$$\partial_k(\sqrt{-g}F_j^{\ k}) \ = \ 0 \ , \tag{1.26}$$

which is related to energy-momentum conservation in general relativity. After some algebra, the Freud pseudo-tensor can be expressed as follows [3, 4, 5],

$$\begin{aligned}\sqrt{-g}F_j^{\ k} \ &= \ 2\left[\sqrt{-g}\,\Box\,\phi_j^{\ k} - \partial_l\Big(\,\mathbf{g}^{mk}\partial_m\phi_j^{\ l} + (\delta_p^{\ k}\delta_j^{\ l} - \delta_p^{\ l}\delta_j^{\ k})\mathbf{g}^{mr}\partial_m\phi_r^{\ p}\,\Big)\right] \\ &= \ 2\sqrt{-g}\,\Box\,\phi_j^{\ k} \ + \text{gauge terms} \ ,\end{aligned} \tag{1.27}$$

with curved spacetime d'Alembertian defined as,

$$\sqrt{-g}\,\Box \ \equiv \ \partial_l(\mathbf{g}^{lm}\partial_m) \ . \tag{1.28}$$

Defining now the mixed indices pseudo-tensor current $j_j^{\ k}$ as follows,

$$\frac{8\pi k}{c^4}\sqrt{-g}\,j_j^{\ k} \ \equiv \ \sqrt{-g}(\,G_j^{\ k} + E_j^{\ k}\,) \ , \tag{1.29}$$

we arrive at the following relationship,

$$\sqrt{-g}F_j^{\ k} \ = \ \frac{8\pi k}{c^4}\sqrt{-g}\,j_j^{\ k} \ . \tag{1.30}$$

Recalling the Freud identity (1.26), we find the current conservation law,

$$\partial_k(\sqrt{-g}\,j_j^{\ k}) \ = \ 0 \ . \tag{1.31}$$

Setting the gauge terms to zero (Newton gauge) in eq. (1.27), the field equation becomes,

$$\sqrt{-g}\,\Box\,\phi_j^{\ k} \ = \ \frac{4\pi k}{c^4}\sqrt{-g}\,j_j^{\ k} \ , \tag{1.32}$$

which is similar to the field equation of electrodynamics. This is the fundamental equation of Einstein general relativity (GR) when the tensor potentials $\phi_j^{\ k}$ are interpreted as the physical gravitational field instead of the metric tensor. This is a direct consequence of the exponential parametrization of the metric.

This formulation of GR is particularly useful because the field equation yields directly the gravitational potential for any type of matter or radiation source. One does not choose a metric *ansatz* for given astrophysical or cosmological problems. One just enters a source term with its physical symmetries in the rhs of the field equation and the corresponding gravitational potential can be calculated in principle from there. Just plug in the desired source and obtain the gravitational potential. The metric and curvature can be calculated from there afterward.

The field equation (1.32) is valid in any coordinate system, whether curvilinear or cartesian. However, as discussed earlier, the use of curvilinear coordinates necessarily creates additional non-tensor forces in the description of the system, the so-called inertial forces, which clutter the dynamics of the real gravitational field. To get a clean view of the real gravitational physics, we are now constraining ourselves to a description in terms of cartesian (natural) coordinates. In such an inertial reference frame, the inertial potentials and inertial forces vanish.

In such a reference frame, the tensor potentials are given by the real gravitational potentials ($\phi_i^j = \varphi_i^j$) and the Christoffel symbols given by the Gravitational symbols ($\Gamma^i{}_{kl} = \Delta^i{}_{kl}$) which are true tensors [3, 4, 5, 6, 7]. Einstein general relativity then becomes an entirely *localized* (no pseudo-tensors) affine tensor theory described as follows [3, 4, 5],

$$G_j^{\,k} = \frac{8\pi k}{c^4}\,\tau_j^{(\mathrm{E})k} \quad ; \quad E_j^{\,k} = \frac{8\pi k}{c^4}\,t_j^{(\mathrm{E})k} \quad ; \quad j_j^{\,k} = \tau_j^{(\mathrm{E})k} + t_j^{(\mathrm{E})k} \ , \qquad (1.33)$$

with matter $\tau_j^{(\mathrm{E})k}$ and real gravitational $t_j^{(\mathrm{E})k}$ energy-momentum (affine) tensors.

The field equation (1.32) is then re-written as (in the Newton gauge),

$$\sqrt{-g}\,\Box\,\varphi_j^{\,k} \;=\; \frac{4\pi k}{c^4}\,\sqrt{-g}\,(\tau_j^{(\mathrm{E})k} + t_j^{(\mathrm{E})k}) \quad ; \quad \partial_k[\sqrt{-g}(\tau_j^{(\mathrm{E})k} + t_j^{(\mathrm{E})k})] \;=\; 0 \ , \qquad (1.34)$$

in which, in cartesian coordinates, the gravitational energy-momentum tensor is given as,

$$\sqrt{-g}\,t_j^{(\mathrm{E})k} \;=\; \frac{c^4}{8\pi k}\Big[-8\,\mathbf{g}^{ks}\,\big(\,\partial_j\varphi_l^{\;m}\partial_m\varphi_s^{\;l} - \frac{1}{2}\partial_j\varphi_l^{\;m}\partial_s\varphi_m^{\;l} + \frac{1}{4}\partial_j\varphi\partial_s\varphi\,\big) + 4\,\delta_j^k\,\mathbf{g}^{rs}\big(\,\partial_l\varphi_r^{\;m}\partial_m\varphi_s^{\;l} - \frac{1}{2}\partial_r\varphi_l^{\;m}\partial_s\varphi_m^{\;l} + \frac{1}{4}\partial_r\varphi\partial_s\varphi\,\big)\,\Big] \ . \qquad (1.35)$$

Let us now consider the general case of an imperfect fluid source described by the following matter energy-momentum tensor [3, 4],

$$\sqrt{-g}\,\tau_j^{(\mathrm{E})k} = \sqrt{-g}\,[\,\mu_0 c^2\,u_j u^k + p_j^{\,k}\,] \ , \qquad (1.36)$$

with proper pressure density stress-tensor $\sqrt{-g}\,p_j^{\,k}$ ($p_0^{\,0} = p_\alpha^{\,0} = p_0^{\,\beta} = 0\,; p_\alpha^{\,\beta} \neq 0$), proper mass density $\sqrt{-g}\,\mu_0$ and 4-velocity u^i.

In a constant and static gravitational field characterized by $g_{ik} \neq g_{ik}(x^0)$ and $g_{0\alpha} = 0$, the 4-velocity u^i is given as follows [1, 2, 3, 4, 5],

$$u^0 = \frac{1}{\sqrt{-g_{00}}\sqrt{1 - v^2/c^2}} \ ; \ u^\alpha = \frac{v^\alpha}{c\sqrt{1 - v^2/c^2}} \ ; \ u_k u^k = -1 \ , \qquad (1.37)$$

with $v^2 = v_\alpha v^\alpha$ and $v^\alpha = \frac{c}{\sqrt{-g_{00}}}\frac{dx^\alpha}{dx^0}$.

We are interested in a slow moving fluid ($\frac{|v|}{c} \ll 1$) and so, at lowest order in velocity expansion, we have,

$$\varphi_j^{\,k} = -\frac{\Phi}{c^2}\,\delta_j^{\,0}\delta_0^{\,k} \quad ; \quad \partial_k\varphi_j^{\,k} = 0\,, \tag{1.38}$$

with time-independent Newton potential Φ. This leads to the following expression for the rescaled metric tensor at the lowest order of the slow velocity expansion,

$$\mathbf{g}^{00} = -e^{-4\Phi/c^2} \;\; ; \;\; \mathbf{g}^{\alpha\beta} = \eta^{\alpha\beta} \;\; ; \;\; \sqrt{-g} = e^{-2\Phi/c^2}\,. \tag{1.39}$$

Similarly, the curved spacetime d'Alembertian acting on a time-independent gravitational field becomes [3, 4, 5],

$$\sqrt{-g}\,\Box \;\rightarrow\; \eta^{\alpha\beta}\partial_\alpha\partial_\beta \;=\; \eta^{\alpha\beta}\nabla_\alpha\nabla_\beta \;\equiv\; \nabla_\alpha\nabla^\alpha \;=\; \vec{\nabla}\cdot\vec{\nabla} \;=\; \vec{\nabla}^2\,, \tag{1.40}$$

and to the same approximation, the Einstein gravitational energy-momentum affine tensor is evaluated as,

$$\begin{aligned} \sqrt{-g}\,t_0^{(\mathrm{E})0} &= -\frac{1}{8\pi k}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi \\ \sqrt{-g}\,t_\alpha^{(\mathrm{E})0} &= \sqrt{-g}\,t_0^{(\mathrm{E})\beta} = 0 \\ \sqrt{-g}\,t_\alpha^{(\mathrm{E})\beta} &= \frac{1}{4\pi k}\,(\nabla_\alpha\Phi\nabla^\beta\Phi - \frac{1}{2}\delta_\alpha^{\,\beta}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi)\,, \end{aligned} \tag{1.41}$$

in which we **defined** the contravariant nabla derivative as $\nabla^\beta \equiv \eta^{\alpha\beta}\nabla_\alpha$. However we still have $\partial^\beta \equiv g^{\alpha\beta}\partial_\alpha$. Therefore although $\partial_\alpha = \nabla_\alpha$ we must remind ourselves that $\partial^\beta \neq \nabla^\beta$.

So, to lowest order in the velocity expansion, the fundamental field equation (1.32) of general relativity yields the following non-trivial relationships [3, 4, 5],

$$\begin{aligned} \vec{\nabla}^2\Phi &= \frac{4\pi k}{c^2}\,[\,\sqrt{-g}\,\mu_0 c^2 + \frac{1}{8\pi k}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi\,]\,, \\ 0 &= \frac{4\pi k}{c^4}\,\big[\,\sqrt{-g}\,p_\alpha^{\,\beta} + \frac{1}{4\pi k}(\nabla_\alpha\Phi\nabla^\beta\Phi - \frac{1}{2}\delta_\alpha^{\,\beta}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi)\,\big]\,. \end{aligned} \tag{1.42}$$

In the second row, we immediately recognize the newtonian gravitational stress tensor $t_\alpha^{(\mathrm{N})\beta}$ [3, 4, 5],

$$\sqrt{-g}\,t_\alpha^{(\mathrm{N})\beta} = \sqrt{-g}\,t_\alpha^{(\mathrm{E})\beta} = \frac{1}{4\pi k}\,(\nabla_\alpha\Phi\nabla^\beta\Phi - \frac{1}{2}\delta_\alpha^{\,\beta}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi)\,, \tag{1.43}$$

which is the spatial part of the Einstein-Pauli gravitational energy-momentum tensor.

The second row of eq. (1.42) yields the following relationships,

$$\begin{aligned} \sqrt{-g}\,p_\alpha^{\,\beta} &= \sqrt{-g}\,\tau_\alpha^{(\mathrm{E})\beta} = -\frac{1}{4\pi k}(\nabla_\alpha\Phi\nabla^\beta\Phi - \frac{1}{2}\delta_\alpha^{\,\beta}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi)\,, \\ \sqrt{-g}\,p_\alpha^{\,\alpha} &= \sqrt{-g}\,\tau_\alpha^{(\mathrm{E})\alpha} = \frac{1}{8\pi k}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi = -\sqrt{-g}\,t_0^{(\mathrm{E})0}\,. \end{aligned} \tag{1.44}$$

Therefore the trace of the pressure density stress tensor $\sqrt{-g}\,p_\alpha^{\,\alpha}$, which is the trace of the spatial part of the Einstein matter energy-momentum tensor

$\sqrt{-g}\,\tau_\alpha^{(E)\alpha}$, is given by the negative of the Einstein-Pauli gravitational energy density $-\sqrt{-g}\,t_0^{(E)0}$.

Inserting this result into the first row of eq. (1.42), we find the following most interesting results,

$$\begin{aligned} \vec{\nabla}^2\Phi &= \frac{4\pi k}{c^2}\sqrt{-g}\,[\,\mu_0 c^2 + p_\alpha^{\ \alpha}\,]\ , \\ &= \frac{4\pi k}{c^2}\sqrt{-g}\,[\,\tau_\alpha^{(E)\alpha} - \tau_0^{(E)0}\,]\ , \end{aligned} \tag{1.45}$$

which implies that the source of the newtonian gravitational potential is really the *total* energy density given by the sum of the matter energy density with the gravitational energy density, itself evaluated solely from the trace of the spatial part of the matter energy-momentum tensor. So the total energy density of matter with gravity can be evaluated with only the knowledge of the matter energy-momentum tensor, a result first obtained by Tolman in 1930 [1, 2].

For a perfect fluid, we have $p_\alpha^{\ \alpha} = 3p$ with p being the proper pressure density of the fluid. Eq. (1.45) is then re-written as [1],

$$\vec{\nabla}^2\Phi = \frac{4\pi k}{c^2}\sqrt{-g}\,[\,\mu_0 c^2 + 3\,p\,]\ . \tag{1.46}$$

Defining the Yilmaz mass and pressure tensor densities as follows,

$$\begin{aligned} \sqrt{-g}\,\mu_0^{(Y)}c^2 &= \sqrt{-g}\,\mu_0 c^2 + \frac{1}{8\pi k}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi \;=\; \sqrt{-g}\,[\,\mu_0 c^2 + p_\alpha^{\ \alpha}\,]\ , \\ \sqrt{-g}\,p_\alpha^{(Y)\beta} &= \sqrt{-g}\,p_\alpha^{\ \beta} + \frac{1}{4\pi k}(\nabla_\alpha\Phi\nabla^\beta\Phi - \frac{1}{2}\delta_\alpha^{\ \beta}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi)\ , \end{aligned} \tag{1.47}$$

the relations (1.42) are re-written as follows,

$$\begin{aligned} \vec{\nabla}^2\Phi &= \frac{4\pi k}{c^2}\sqrt{-g}\,\mu_0^{(Y)}c^2 \;=\; -\frac{4\pi k}{c^2}\sqrt{-g}\,\tau_0^{(Y)0} \\ 0 &= \sqrt{-g}\,p_\alpha^{(Y)\beta} \;=\; \sqrt{-g}\,\tau_\alpha^{(Y)\beta}\ , \end{aligned} \tag{1.48}$$

which are nothing but the Yilmaz gravitational field equations [3, 4, 5, 9]. Misner [19] correctly stated that the Yilmaz theory led to a zero pressure theory, but he did not realize that the expressions (1.48) included both the matter and gravitational energy-momentum tensor contributions. Alley, Aschan and Yilmaz [20], in their refutation of Misner's article, correctly emphasized this crucial point.

In fact, as noted previously by the author [3, 4], Yilmaz and Einstein general relativity are completely equivalent and are related to each other by the following relationships,

$$\begin{aligned} \tau_j^{(Y)k} &= \tau_j^{(E)k} \;+\; t_j^{(E)k} \\ t_j^{(Y)k} &= -t_j^{(E)k}\ . \end{aligned} \tag{1.49}$$

An equivalence which can only be held when making use of the Yilmaz exponential parametrization of the metric tensor in both theories.

2 Gravitational Energy and Hydrostatic Equilibrium

Several decades ago, past authors [1, 2, 21, 22] have tackled the problem of gravitational energy and total gravitational mass defect in the context of a spherically symmetric constant and static gravitational field ($g_{ik} \neq g_{ik}(x^0)$; $g_{0\alpha} = 0$). In the traditional formulation of Einstein general relativity (GR), this relates to the Schwarzschild solution.

Tolman [2], Zel'dovich and Novikov [21] as well as Landau and Lifchitz [1] defined the following conserved **total energy** E of matter and gravitational field as follows,

$$\begin{aligned} cP^0 \;=\; E \;&=\; E^{(\mathrm{ZN})} \;=\; M^{(\mathrm{ZN})}c^2 \;\equiv\; M^{(\mathrm{ZN})}(R)\,c^2 \\ E^{(\mathrm{ZN})} \;&\equiv\; -\int dV\sqrt{-g}\,[\,\tau_0^{(\mathrm{E})0} + t_0^{(\mathrm{E})0}\,] \\ &=\; \int_V dV\sqrt{-g}\,[\,\tau_\alpha^{(\mathrm{E})\alpha} - \tau_0^{(\mathrm{E})0}\,] \\ M^{(\mathrm{ZN})}(r)\,c^2 \;&\equiv\; \int_0^r 4\pi r^2 dr\sqrt{-g(r)}\,[\,\mu_0(r)c^2 + 3\,p(r)\,]\;, \end{aligned} \tag{2.1}$$

in which $dV = 4\pi r^2 dr$ and (V,R) are respectively the volume and radius of the spherically symmetric perfect fluid star interior and $M^{(\mathrm{ZN})}$ its **total rest mass**, which includes the gravitational binding energy of its constituents.

The total energy $E^{(\mathrm{S})}$, in the context of the *Schwarzschild metric*, is equivalently given as [1, 2, 21, 22],

$$\begin{aligned} E \;=\; E^{(\mathrm{S})} \;=\; M^{(\mathrm{S})}c^2 \;&\equiv\; M^{(\mathrm{S})}(R)\,c^2 \;=\; E^{(\mathrm{ZN})} \\ M^{(\mathrm{S})}(r)\,c^2 \;&\equiv\; \int_0^r 4\pi r^2 dr\,\mu_0(r)c^2\;, \end{aligned} \tag{2.2}$$

an equality which is surprising considering the fact that the gravitational energy is supposed to be included in the expression. But calculation shows it to be true for the Schwarzschild metric.

According to Weinberg [22] on the other hand, the **matter energy alone** (total energy without the gravitational binding energy) is not really well defined. Weinberg offers us the following definition [22],

$$E_1^{(\mathrm{W})} \;=\; M_1^{(\mathrm{W})}c^2 \;\equiv\; \int_0^R 4\pi r^2 dr\,\sqrt{-g(r)}\,\mu_0(r)c^2\;. \tag{2.3}$$

But given the reputed equivalence $E^{(\mathrm{S})} = E^{(\mathrm{ZN})}$ [1, 2, 21, 22] between the Schwarzschild and the Zel'dovich-Novikov formulae (2.1)-(2.2) for the total energy, we therefore arrive at the following result for the gravitational energy,

$$E_G^{(\mathrm{W})} \;\equiv\; E^{(\mathrm{S})} - E_1^{(\mathrm{W})} \;=(?)\; E^{(\mathrm{ZN})} - E_1^{(\mathrm{W})} \;>\; 0\;. \tag{2.4}$$

which is *unacceptable* since the gravitational energy should really be a *negative* binding energy.

Gravitational Energy in Einstein Real Gravity

Let us therefore revisit these considerations in the context of Einstein Real Gravity [3, 4, 5, 6]. In this context obviously, the Schwarzschild expression (2.2) for the total energy is no longer valid. Recalling the field equation (1.46), we see that what the many authors called the total matter energy given by the Tolman formula (2.1) is really the source of the gravitational field. However this includes the gravitational energy contribution given by the trace of the Newton gravitational stress tensor with value proportional to $+3p$. This gravitational energy has the *wrong sign* and leads to the problem discussed earlier with Weinberg's expressions (2.3) for the matter energy alone.

Physically, as already discussed by Alley, Aschan and Yilmaz [20], the Einstein energy-momentum tensor $\tau_j^{(\mathrm{E})k}$ in eq. (1.45) ought to describe the *total* energy-momentum of matter bound by gravity, *i.e* including its gravitational binding energy. But the gravitational binding energy of an astronomical body *cannot* belong to the source of the gravitational field the body generates, because otherwise gravity would be self-generating. So we must subtract to the total energy density of the body its gravitational binding energy. Since this energy density is negative and proportional to $-3p$, the internal pressure of the body, subtracting the binding energy is same as adding up a term proportional to $+3p$ in the source term of the gravitational field. Recalling the relationships (1.49) between the Einstein and Yilmaz energy-momentum tensors, we therefore find that it is the Yilmaz matter density (1.47) which is the source of the gravitational potential, *i.e* the matter density alone.

Since gravity should not be self-generating, we are then led to the following identification of the **matter energy alone** in Real Gravity (RG),

$$\begin{aligned} E_1^{(\mathrm{RG})} \;=\; M_1^{(\mathrm{RG})}c^2 \;&\equiv\; -\int dV\sqrt{-g}\,[\,\tau_0^{(\mathrm{E})0} + t_0^{(\mathrm{E})0}\,] \\ &=\; -\int dV\sqrt{-g}\,\tau_0^{(\mathrm{Y})0} \\ &=\; \int_V dV\sqrt{-g}\,[\,\mu_0 c^2 + 3\,p\,] \\ &=\; E^{(\mathrm{ZN})}\;. \end{aligned} \tag{2.5}$$

Recalling again the relationships (1.49), we then identify the **total energy** of matter and gravitation in Real Gravity (RG) as follows,

$$\begin{aligned} E^{(\mathrm{RG})} \;=\; M^{(\mathrm{RG})}c^2 \;&=\; -\int dV\sqrt{-g}\,\tau_0^{(\mathrm{E})0} \\ &=\; -\int dV\sqrt{-g}\,[\,\tau_0^{(\mathrm{Y})0} + t_0^{(\mathrm{Y})0}\,] \\ &=\; \int_V dV\sqrt{-g}\,\mu_0 c^2 \\ &=\; E_1^{(\mathrm{W})}\;. \end{aligned} \tag{2.6}$$

The **gravitational binding energy** in Real Gravity (RG) is then simply given by,

$$E_G^{(\mathrm{RG})} \;=\; M_G^{(\mathrm{RG})}c^2 \;=\; E^{(\mathrm{RG})} - E_1^{(\mathrm{RG})}$$

$$= \; - \int dV \sqrt{-g}\, t_0^{(\mathrm{Y})0} \; = \; -3 \int_V dV \sqrt{-g}\, p \; , \tag{2.7}$$

which corresponds to the Newtonian value [23] and which has the right *negative sign*, as it should be for a binding energy.

So the total energy $E^{(\mathrm{RG})}$ is obtained by the *Einstein* energy-momentum tensor while the gravitational energy $E_G^{(\mathrm{RG})}$ is related to the internal matter pressure. The matter energy alone $E_1^{(\mathrm{RG})}$ is then simply given by the difference of both quantities $E_1^{(\mathrm{RG})} = E^{(\mathrm{RG})} - E_G^{(\mathrm{RG})}$, which is the Tolman formula (2.1), and is the *sole physical source* of the gravitational field Φ according to eqs. (1.45) or (1.48). This is very different from all the previous considerations [1, 2, 21, 22] for which the Tolman formula gives the total energy. In Einstein Real Gravity, the Tolman formula yields the matter energy alone, without its gravitational binding energy, which has been subtracted off by the presence of the term proportional to $+3p$.

The Hydrostatic Equilibrium Formula

Recalling the relationships (1.42)-(1.45) between the gravitational potential, the Einstein mass density and pressure tensor, and making use of the Freud identity (1.34),

$$\nabla_\alpha[\sqrt{-g}(\tau_\beta^{(\mathrm{E})\alpha} + t_\beta^{(\mathrm{E})\alpha})] \; = \; 0 \; , \tag{2.8}$$

we easily derive the following relation,

$$\begin{aligned} \nabla_\alpha(\sqrt{-g}\, p_\beta^\alpha) \; &= \; -\nabla_\alpha(\sqrt{-g}\, t_\beta^{(\mathrm{E})\alpha}) \\ &= \; -\frac{1}{4\pi k}\,(\vec{\nabla}^2\Phi)\,\nabla_\beta\Phi \\ &= \; -\frac{\sqrt{-g}}{c^2}\,[\,\mu_0 c^2 + p_\alpha^{\;\alpha}\,]\,\nabla_\beta\Phi \;\; . \end{aligned} \tag{2.9}$$

Recalling that,

$$\nabla_\alpha \ln\sqrt{-g} \; = \; -\frac{2}{c^2}\,\nabla_\alpha\Phi \; , \tag{2.10}$$

the expression (2.9) for the Freud identity finally becomes,

$$\nabla_\alpha p_\beta^{\;\alpha} \; = \; -\frac{1}{c^2}\left[\,(\,\mu_0 c^2 + p_\alpha^{\;\alpha}\,)\,\nabla_\beta\Phi \; - \; 2\,p_\beta^{\;\alpha}\nabla_\alpha\Phi\right] \;\; . \tag{2.11}$$

The above equation establishes the condition for the gravitational equilibrium of an astronomical body of given mass density and internal pressure stress tensor in the context of Einstein Real Gravity (RG).

Restricting ourselves to the case of a spherically symmetric perfect fluid in cartesian coordinates, we have $p_\beta^{\;\alpha} = p\delta_\beta^{\;\alpha}$ and $p_\alpha^{\;\alpha} = 3p$. The equilibrium equation (2.11) then simply becomes,

$$\nabla_\alpha p \; = \; -(\,\mu_0 + p/c^2\,)\,\nabla_\alpha\Phi \;\; . \tag{2.12}$$

In the case of spherical symmetry ($r = \sqrt{x^2+y^2+z^2}$), the above equation can be re-written as follows,

$$r^2\partial_r p(r) \; = \; -[\,\mu_0(r) + p(r)/c^2\,]\; r^2\partial_r\Phi(r) \;\; , \tag{2.13}$$

and the field equation (1.46) for the gravitational potential $\Phi(r)$ becomes,

$$\begin{aligned} r^2\vec{\nabla}^2\Phi &= \partial_r[r^2\partial_r\Phi(r)] \\ &= \frac{k}{c^2}\,4\pi r^2\sqrt{-g(r)}\,[\,\mu_0(r)c^2 + 3p(r)\,] \\ &= \frac{k}{c^2}\,\partial_r[\,M^{(\mathrm{RG})}(r)c^2 + 3P^{(\mathrm{RG})}(r)\,]\ , \end{aligned} \tag{2.14}$$

leading to,

$$r^2\partial_r\Phi(r) = \frac{k}{c^2}\,[\,M^{(\mathrm{RG})}(r)c^2 + 3P^{(\mathrm{RG})}(r)\,]\ , \tag{2.15}$$

in which we defined,

$$\begin{aligned} M^{(\mathrm{RG})}(r)\,c^2 &\equiv \int_0^r 4\pi r^2 dr\sqrt{-g(r)}\,\mu_0(r)c^2 \\ P^{(\mathrm{RG})}(r) &\equiv \int_0^r 4\pi r^2 dr\sqrt{-g(r)}\,p(r)\ , \end{aligned} \tag{2.16}$$

and so,

$$\begin{aligned} \partial_r M^{(\mathrm{RG})}(r)\,c^2 &= 4\pi r^2\sqrt{-g(r)}\,\mu_0(r)c^2 \\ \partial_r P^{(\mathrm{RG})}(r) &= 4\pi r^2\sqrt{-g(r)}\,p(r)\ . \end{aligned} \tag{2.17}$$

Recalling eqs. (2.1) and (2.5), we take note of the following relationships,

$$\begin{aligned} M_1^{(\mathrm{RG})}(r)\,c^2 &\equiv M^{(\mathrm{RG})}(r)\,c^2 + 3P^{(\mathrm{RG})}(r) = M^{(\mathrm{ZN})}(r)\,c^2 \\ M_1^{(\mathrm{RG})}(R)\,c^2 &= M^{(\mathrm{RG})}(R)\,c^2 + 3P^{(\mathrm{RG})}(R) \\ &= M_1^{(\mathrm{RG})}\,c^2 = E_1^{(\mathrm{RG})} = E^{(\mathrm{ZN})}\ , \end{aligned} \tag{2.18}$$

which are expressions for the **matter energy alone** in Einstein Real Gravity. Recalling now eq. (2.6), we further get,

$$M^{(\mathrm{RG})}(R)\,c^2 = M^{(\mathrm{RG})}\,c^2 = E^{(\mathrm{RG})} = E_1^{(\mathrm{W})}\ , \tag{2.19}$$

for the **total energy**, which immediately leads to the following expression for the **gravitational binding energy** in Einstein Real Gravity,

$$E_G^{(\mathrm{RG})} = E^{(\mathrm{RG})} - E_1^{(\mathrm{RG})} = -3P^{(\mathrm{RG})}(R)\ . \tag{2.20}$$

Inserting now eqs. (2.15)-(2.16) into eq. (2.13) finally yields the sought after relationship,

$$\begin{aligned} \partial_r p(r) &= -\frac{k\mu_0(r)M^{(\mathrm{RG})}(r)}{r^2}\Big[1+\frac{p(r)}{\mu_0(r)c^2}\Big]\Big[1+\frac{3P^{(\mathrm{RG})}(r)}{M^{(\mathrm{RG})}(r)c^2}\Big] \\ &= -\frac{k\mu_0(r)M_1^{(\mathrm{RG})}(r)}{r^2}\Big[1+\frac{p(r)}{\mu_0(r)c^2}\Big]\ . \end{aligned} \tag{2.21}$$

This is the fundamental **hydrostatic equilibrium formula** of Einstein Real Gravity (RG), to be compared with the so-called **Tolman-Oppenheimer-Volkoff equation** for the Schwarzschild metric [21, 22],

$$\begin{aligned} \partial_r p(r) &= -\frac{k\mu_0(r)M^{(\mathrm{S})}(r)}{r^2}\Big[1+\frac{p(r)}{\mu_0(r)c^2}\Big]\Big[1+\frac{3P^{(\mathrm{S})}(r)}{M^{(\mathrm{S})}(r)c^2}\Big] \\ &\quad\times\Big[1-\frac{2kM^{(\mathrm{S})}(r)}{rc^2}\Big]^{-1}\ , \end{aligned} \tag{2.22}$$

where we defined,

$$P^{(\mathrm{S})}(r) \;\equiv\; V(r)\,p(r) \quad ; \quad V(r) \;\equiv\; \frac{4\pi r^3}{3} \quad . \tag{2.23}$$

Although similar, the relationships (2.21) and (2.22) differ by the definition of $P(r)$ and $M(r)$, as well as the Schwarzschild factor $\left[\,1 - 2kM^{(\mathrm{S})}(r)/rc^2\,\right]^{-1}$ in equation (2.22). The new formula (2.21) does not have singularity problems at the Schwarzschild radius.

Both formulae lead to the same Newtonian limit $p(r) \ll \mu_0(r)c^2$,

$$\partial_r p(r) \;\to\; -\frac{k\mu_0(r)M_1(r)}{r^2} \;\simeq\; -\frac{k\mu_0(r)M(r)}{r^2} \quad . \tag{2.24}$$

with $M(r)$ the total mass of the star (or astronomical body) at radius r.

Various Energy Definitions in Einstein Real Gravity

We give here a list of expressions of the various energies and quantities defined in the context of Einstein Real Gravity.

Total energy $E^{(\mathrm{RG})}$ of matter and gravitation :

$$\begin{aligned} E^{(\mathrm{RG})} \;=\; M^{(\mathrm{RG})}c^2 \;&\equiv\; -\int dV\sqrt{-g}\,\tau_0^{(\mathrm{E})0} \\ &=\; -\int dV\sqrt{-g}\,[\,\tau_0^{(\mathrm{Y})0} + t_0^{(\mathrm{Y})0}\,] \\ &=\; \int_V dV\sqrt{-g}\,\mu_0 c^2 \\ &=\; \int_0^R 4\pi r^2 dr\sqrt{-g(r)}\,\mu_0(r)c^2\;. \end{aligned} \tag{2.25}$$

Matter energy alone $E_1^{(\mathrm{RG})}$ (Tolman relation [1, 2, 21]) :

$$\begin{aligned} E_1^{(\mathrm{RG})} \;=\; M_1^{(\mathrm{RG})}c^2 \;&\equiv\; -\int dV\sqrt{-g}\,[\,\tau_0^{(\mathrm{E})0} + t_0^{(\mathrm{E})0}\,] \\ &=\; -\int dV\sqrt{-g}\,\tau_0^{(\mathrm{Y})0} \\ &=\; \int_V dV\sqrt{-g}\,[\,\mu_0 c^2 + 3\,p\,] \\ &=\; \int_0^R 4\pi r^2 dr\sqrt{-g(r)}\,[\,\mu_0(r)c^2 + 3\,p(r)\,]\;. \end{aligned} \tag{2.26}$$

Gravitational binding energy $E_G^{(\mathrm{RG})}$:

$$E_G^{(\mathrm{RG})} \;=\; M_G^{(\mathrm{RG})}c^2 \;\equiv\; -\int dV\sqrt{-g}\,t_0^{(\mathrm{Y})0}$$

$$
\begin{aligned}
&= -3 \int_V dV \sqrt{-g}\, p \\
&= -3 \int_0^R 4\pi r^2 dr \sqrt{-g(r)}\, p(r) \;. \qquad (2.27)
\end{aligned}
$$

Total particle number $N^{(\mathrm{RG})}$:

$$
\begin{aligned}
N^{(\mathrm{RG})} &\equiv \int_V dV \sqrt{-g}\, \nu_0 \\
&= \int_0^R 4\pi r^2 dr \sqrt{-g(r)}\, \nu_0(r) \;, \qquad (2.28)
\end{aligned}
$$

with *proper particle number density* $\nu_0(r)$.

Sole rest mass energy $E_0^{(\mathrm{RG})}$:

$$
\begin{aligned}
E_0^{(\mathrm{RG})} &\equiv m_p c^2 \int_V dV \sqrt{-g}\, \nu_0 \\
&= N^{(\mathrm{RG})} m_p c^2 \;, \qquad (2.29)
\end{aligned}
$$

with *individual particle rest mass* m_p.

Thermal energy $E_\beta^{(\mathrm{RG})}$:

$$
\begin{aligned}
E_\beta^{(\mathrm{RG})} &\equiv \int_V dV \sqrt{-g}\, e_0 \\
&= \int_0^R 4\pi r^2 dr \sqrt{-g(r)}\, e_0(r) \\
e_0(r) &\equiv [\, \mu_0(r) c^2 + 3\, p(r)\,] - m_p c^2 \nu_0(r) \;, \qquad (2.30)
\end{aligned}
$$

with *proper internal energy density* $e_0(r)$.

Internal energy $W^{(\mathrm{RG})}$:

$$
\begin{aligned}
W^{(\mathrm{RG})} &\equiv \int_V dV \sqrt{-g}\, [\, e_0 - 3p\,] \\
&= \int_0^R 4\pi r^2 dr \sqrt{-g(r)}\, [\, e_0(r) - 3\, p(r)\,] \;. \qquad (2.31)
\end{aligned}
$$

From the above definitions, we arrive at the following relationships among the various quantities,

$$
\begin{aligned}
W^{(\mathrm{RG})} &= E_\beta^{(\mathrm{RG})} + E_G^{(\mathrm{RG})} = E^{(\mathrm{RG})} - E_0^{(\mathrm{RG})} \\
E_\beta^{(\mathrm{RG})} &= E^{(\mathrm{RG})} - E_0^{(\mathrm{RG})} - E_G^{(\mathrm{RG})} = E_1^{(\mathrm{RG})} - E_0^{(\mathrm{RG})} \\
E_1^{(\mathrm{RG})} &= E_\beta^{(\mathrm{RG})} + E_0^{(\mathrm{RG})} \\
E^{(\mathrm{RG})} &= E_1^{(\mathrm{RG})} + E_G^{(\mathrm{RG})} = E_0^{(\mathrm{RG})} + W^{(\mathrm{RG})} \;. \qquad (2.32)
\end{aligned}
$$

3 Ultra-Relativistic Spherical Fluid

Let us go back to the relationship (2.9) between the pressure and the gravitational potential of a time-independent fluid source. For a perfect fluid in cartesian coordinates we have ($p^{\alpha}_{\beta} = p\delta^{\alpha}_{\beta}$; $p^{\alpha}_{\alpha} = 3p$),

$$\vec{\nabla}(\sqrt{-g}\,p) \;=\; -\sqrt{-g}\,[\,\mu_0 c^2 + 3p\,]\,\vec{\nabla}\left(\frac{\Phi}{c^2}\right) \;. \qquad (3.1)$$

Assuming the spherical symmetry of the fluid body, the above relationship is re-written as follows,

$$r^2\partial_r(\sqrt{-g}\,p) \;=\; -\sqrt{-g}\,[\,\mu_0 c^2 + 3p\,]\,r^2\partial_r\left(\frac{\Phi}{c^2}\right) \;. \qquad (3.2)$$

Recalling the relationship (2.15) between the spherically symmetric gravitational potential and its matter source, the above eq. (3.2) leads directly to the following workable expression for the hydrostatic equilibrium formula,

$$r^2\partial_r[\sqrt{-g}(r)\,p(r)] \;=\; -\frac{k}{4\pi r^2}\,M_1^{(\mathrm{RG})}(r)\,\partial_r M_1^{(\mathrm{RG})}(r) \;, \qquad (3.3)$$

a form in agreement with the Newtonian expression for the gravitational energy [23].

For an ultra-relativistic perfect fluid, for instance a cold high-density neutron star [21, 22], we write the following equation of state,

$$p \;=\; \frac{\mu_0 c^2}{3} \;, \qquad (3.4)$$

which however, in the case of a neutron star, should not remain valid as we approach the surface of the star since the neutrons are believed to be non-relativistic there. Nevertheless, assuming the full validity of the above ultra-relativistic equation of state, we now derive the exact analytical solution to the hydrostatic equilibrium formula (3.3).

Dimensional inspection motivates the following *ansatz*,

$$\sqrt{-g}(r)\,\mu_0(r)c^2 \;=\; K\left(\frac{r_0}{r}\right)^2 \;. \qquad (3.5)$$

Inserting this *ansatz* into the hydrostatic equilibrium formula (3.3) and making use of the equation of state (3.4) yields the following solution,

$$K r_0^2 \;=\; \frac{c^4}{24\pi k} \;, \qquad (3.6)$$

in which use was made of the relationships (2.16)-(2.18). Therefore we find straightforwardly,

$$\sqrt{-g(r)}\,\mu_0(r)c^2 \;=\; \frac{c^4}{24\pi k r^2} \quad ; \quad \sqrt{-g(r)}\,p(r) \;=\; \frac{c^4}{72\pi k r^2} \;. \qquad (3.7)$$

Let us now compute the gravitational potential $\Phi(r)/c^2$. Recalling again the relationship (2.16) and making use of the solutions (3.7), we arrive at the following expressions,

$$
\begin{aligned}
M^{(\mathrm{RG})}(r)\,c^2 &= \frac{c^4}{24\pi k}\int_0^r 4\pi r^2 dr\left(\frac{1}{r^2}\right) = \frac{c^4 r}{6k} \\
P^{(\mathrm{RG})}(r) &= \frac{M^{(\mathrm{RG})}(r)\,c^2}{3} = \frac{c^4 r}{18k}\ . \qquad (3.8)
\end{aligned}
$$

Inserting these expressions into eq. (2.15) for the gravitational potential yields for $r > 0$,

$$
\frac{\Phi(r)}{c^2} = -\frac{1}{3}\ln\left(\frac{r_0}{r}\right) - \frac{R_0}{r}\ , \qquad (3.9)
$$

with free radii parameters r_0 and R_0 to be determined from certain boundary conditions. Note that the second term with R_0 in eq. (3.9) is the homogeneous part given by the solution of the Laplace equation $\vec{\nabla}^2\Phi = 0$. Now since,

$$
\sqrt{-g(r)} = e^{-2\Phi(r)/c^2}\ \ ;\ \ \sqrt{-g_{00}(r)} = e^{\Phi(r)/c^2}\ , \qquad (3.10)
$$

we finally get $(r > 0)$,

$$
\mu_0(r)c^2 = \frac{c^4\,e^{-2R_0/r}}{24\pi k r_0^2}\left(\frac{r_0}{r}\right)^{4/3}\ ;\ p(r) = \frac{c^4\,e^{-2R_0/r}}{72\pi k r_0^2}\left(\frac{r_0}{r}\right)^{4/3}\ , \qquad (3.11)
$$

which, for finite positive R_0, allow the density and pressure to vanish in the limit $r \to 0$.

On the other hand the *observable* mass and pressure densities are given by the product of the proper densities with the factor $\sqrt{-g_{00}(r)}$. This is so since the invariant spatial volume element in the reference frame of the calculation (the observer) is given by $\frac{\sqrt{-g}}{\sqrt{-g_{00}}}\,dV$ for a static spacetime ($g_{0\alpha} = 0$). Therefore we get $(r > 0)$,

$$
\sqrt{-g_{00}(r)}\,\mu_0(r)c^2 = \frac{c^4\,e^{-3R_0/r}}{24\pi k r_0^2}\left(\frac{r_0}{r}\right)\ ;\ \sqrt{-g_{00}(r)}\,p(r) = \frac{c^4\,e^{-3R_0/r}}{72\pi k r_0^2}\left(\frac{r_0}{r}\right)\ , \qquad (3.12)
$$

which again vanish in the limit $r \to 0$.

The results of eq. (3.11) in terms of the proper mass density $\mu_0(r)$ and proper pressure density $p(r)$ are to be compared with the high-density neutron star results [21, 22] from the old Tolman-Oppenheimer-Volkoff hydrostatic equilibrium formula for the Schwarzschild metric,

$$
\mu_0(r)c^2 = \frac{3c^4}{56\pi k r^2}\ ;\ p(r) = \frac{c^4}{56\pi k r^2}\ , \qquad (3.13)
$$

which yields infinite density and pressure in the limit $r \to 0$.

Let us finally compute the various energies associated with the solution (3.7) or equivalently (3.11) in the context of Einstein Real Gravity. Recalling equations (2.25)-(2.32), we easily find the following expressions,

$$
\begin{aligned}
E^{(\mathrm{RG})} &= E^{(\mathrm{RG})}(R) = \frac{c^4 R}{6k} \\
E_1^{(\mathrm{RG})} &= E^{(\mathrm{RG})}(R) + 3P^{(\mathrm{RG})}(R) = \frac{c^4 R}{3k} = 2E^{(\mathrm{RG})}
\end{aligned}
$$

$$\begin{aligned}
E_G^{(\mathrm{RG})} &= E^{(\mathrm{RG})} - E_1^{(\mathrm{RG})} = -\frac{c^4 R}{6k} = -E^{(\mathrm{RG})} \\
E_0^{(\mathrm{RG})} &= N^{(\mathrm{RG})} m_p c^2 \\
E_\beta^{(\mathrm{RG})} &= E_1^{(\mathrm{RG})} - E_0^{(\mathrm{RG})} = \frac{c^4 R}{3k} - N^{(\mathrm{RG})} m_p c^2 \\
W^{(\mathrm{RG})} &= E^{(\mathrm{RG})} - E_0^{(\mathrm{RG})} = \frac{c^4 R}{6k} - N^{(\mathrm{RG})} m_p c^2 \,,
\end{aligned} \tag{3.14}$$

with R the radius of the ultra-relativistic spherical fluid (star).

4 Concluding Remarks

There are many theoretical arguments supporting the use of the tensor potentials φ_j^k instead of the metric tensor g_{ik} as the physical gravitational field [13, 14, 15, 16]. But in this work, we showed that a modified Tolman-Oppenheimer-Volkoff hydrostatic equation emerges from the Einstein Real Gravity formulation of general relativity, thereby providing a potential way for astrophysicists to determine experimentally which formulation of general relativity agrees best with nature.

Einstein Real Gravity is a formulation of general relativity describing the spacetime curving part of the gravitational force. It is always a pure tensor force residing at the core of general relativity, but surrounded and attached by the equivalence principle to additional non-tensor inertial forces unhappily unified with it.

The use of the natural cartesian coordinates however cleans up the dynamics and truly eliminates the unwanted non-tensor inertial forces, rendering general relativity an affine tensor theory of pure spacetime curving gravity. The pure spacetime curving tensor force is always present in any curvilinear coordinates, but the use of cartesian coordinates liberates it. The tensorial nature of the spacetime curving gravitational force implies that pure gravitational energy-momentum is localized in spacetime, an essential condition for the propagation of gravitational waves.

If this formulation of GR is the physical one, it is particularly efficient because the field equation yields directly the gravitational potential for any type of matter or radiation source. The symmetries of the problem are dictated by the ones of the physical sources. So one does not have to worry about choosing a metric ansatz for given astrophysical or cosmological problems. One just enters a source term corresponding to the physical situation and one gets immediately the gravitational potential, be it a star, a MECO [15], a galaxy or a given matter distribution for the whole universe. No need to invent any science-fiction metric. Just plug in the desired source and get the gravitational potential. The metric and curvature can be calculated from there afterward. But such calculations of course often remain very challenging.

From the viewpoint of quantization, the separation of pure gravitational from inertial forces is a necessity. Only spacetime curving forces need to be quantized, as they solely will describe quantum gravitons. Other inertial (spacetime non-curving) forces such a the Coriolis or centrifugal forces do not necessitate quantization. They are coordinate-dependent classical forces and not propagated by quantum particles.

A thorough analysis of the modified Tolman-Oppenheimer-Volkoff hydrostatic equation found in this paper and its consequences for the stability theory of the various species of stars as well as a study of star evolution and gravitational collapse is work for the future.

Acknowledgements

Much of the work presented in this paper could not have been done without the motivating and illuminating exchanges I was blessed to have in the last few years with the following three eminent astrophysicists: Prof. Stanley L. Robertson (Southwestern Oklahoma State University), Prof. Rudolph E. Schild (Harvard-Smithsonian Center for Astrophysics) and Prof. Abhas Mitra (Homi Bhabha National Institute, Mumbai).

I also benefited from my continuing exchanges with Prof. Vladimir Belinski (ICRA, Rome), a highly distinguished general relativist from the great Landau school in Moscow, who, despite his critics of some of my current views regarding general relativity, remained a precious source of discussions and learning. In the nineties, we both worked independently, him on the problem of the Hawking evaporation effect and me on Black Hole Thermodynamics. Our conclusions converged to the same verdict, *i.e.* the fallacy of the whole Hawking quantum black hole theory. Since then, I even abandoned the idea of the very existence of classical black holes. He did not.

References

[1] L. Landau and E. Lifchitz, *Théorie du Champ*, Physique Théorique Vol. II (Éditions MIR, Moscou, 1966).

[2] Richard C. Tolman, *Relativity, Thermodynamics and Cosmology*, (Dover Publications, New York, 2019).

[3] Y. Leblanc, *"Gravitational Energy and Star Equilibrium in Einstein Real Gravity"*, manuscript no. eFTC-220401 (2022); doi:10.13140/RG.2.2.13418.59843/1.

[4] Y. Leblanc, *"The Classical Theory of Einstein Real Gravity"*, manuscript no. eFTC-220201 (2022); doi:10.13140/RG.2.2.15970.40640.

[5] Y. Leblanc, *"Comparative Analysis and Complete Equivalence of Einstein and Yilmaz General Relativity"*, manuscript no. eFTC-210301 (2021); doi:10.13140/RG.2.2.11373.92640.

[6] Y. Leblanc, *"Energy-Momentum Conservation and the Freud Identity in General Relativity"*, manuscript no. eFTC-210601 (2021); doi:10.13140/RG.2.2.26577.04960.

[7] Y. Leblanc, *"Separating Coordinates from Pure Gravity Effects in Einstein and Yilmaz General Relativity"*, manuscript no. eFTC-200601 (2020); doi:10.13140/RG.2.2.14981.58082.

[8] H. M. Schwartz, *"Einstein's comprehensive 1907 essay on relativity, part III"*, Am. J. Phys. **45**, 899 (1998), doi:10.1119/1.10743.

[9] Huseyin Yilmaz, *"Toward a field theory of gravitation"*, Il Nuovo Cimento B **107**, 941–960 (1992); doi:10.1007/BF02899296.

[10] Huseyin Yilmaz, *"Relativity and quantum mechanics"*, Int. J. of Theor. Phys. **21**, 871–902 (1982); doi:10.1007/BF01856878.

[11] Huseyin Yilmaz, *"Gravity and Quantum Field Theory : A Modern Synthesis"*, Annals of the New York Academy of Sciences **755**, 476–499 (1995); doi:10.1111/j.1749-6632.1995.tb38990.x.

[12] Carroll O. Alley, Darryl Leiter, Yutaka Mizobuchi and Huseyin Yilmaz, *"Energy Crisis in Astrophysics (Black Holes vs. N-Body Metrics)"*, UMCP Physics Department Pre-print 99-PP-113 (1999); arXiv:astro-ph/9906458v1.

[13] Carroll O. Alley, *"Relativistic Time Measurements and Other Experiments Require Going Beyond Einstein to the Yilmaz Theory of Gravity"*, Talk presented at the Science Session of Commission 31, XXVIth IAU General Assembly (Prague, Czech Republic, August 14-25, 2006); url:alley.ppt.

[14] Carroll O. Alley, *"The Yilmaz Theory of Gravity and its Compatibility with Quantum Theory"*, Annals of the New York Academy of Sciences **755**, 464–475 (1995); doi:10.1111/j.1749-6632.1995.tb38989.x.

[15] Stanley L. Robertson, *"MECO In An Exponential Metric"*, (2016); arXiv:1606.01417v1; *and references therein.*

[16] Stanley L. Robertson, *"Yilmaz Theory of SNe 1a Redshift"*, (2015); arXiv:1507.07809v2.

[17] P. Freud, Ann. Math, Princeton **40**, 417 (1939).

[18] W. Pauli, *Theory of Relativity*, (Dover Publications, New York, 1958).

[19] Charles W. Misner, *"Yilmaz Cancels Newton"*, Il Nuovo Cimento B **114**, 1079–1085 (1999); arXiv:gr-qc/9504050v2.

[20] Carroll O. Alley, Per Kenneth Aschan and Huseyin Yilmaz, *"Refutation of C.W. Misner's claims in his article "Yilmaz Cancels Newton""*, arXiv:gr-qc/9506082.

[21] Ya. B. Zel'dovich and I. D. Novikov, *Relativistic Astrophysics*, Vol. 1: Stars and Relativity (University of Chicago Press, Chicago, IL, 1971).

[22] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (John Wiley and Sons, New York, NY, 1972).

[23] L. Landau and E. Lifchitz, *Physique Statistique*, Physique Théorique Vol. V (Éditions MIR, Moscou, 1967).