# Description of a Turbulent Cascade by a Fokker-Planck Equation

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(Received 1 July 1996)

From experimental data of a turbulent free jet we calculate the joint probability distribution $p(v_1, L_1; v_2, L_2)$ for two velocity increments $v_1, v_2$ of different length scales $L_1, L_2$. We present experimental evidence that the conditional probability distribution $p(v_2, L_2 \mid v_1, L_1)$ obeys a Fokker-Planck equation. We calculate the corresponding drift and diffusion coefficients and discuss their relationship to universal behavior in the scaling region and to intermittency of the turbulent cascade. We explicitly present a stochastic process for the log-normal model of Kolmogorov and Oboukhov [A. M. Oboukhov, J. Fluid Mech. **13**, 77 (1962); A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962)]. [S0031-9007(96)02233-8]

PACS numbers: 47.27.Eq, 02.50.Fz, 05.40.+j

Fully developed turbulence is still regarded to be one of the main unsolved problems of classical physics. Great efforts have been made towards an understanding of small scale turbulent velocity fluctuations, which are assumed to be stationary, homogeneous, and isotropic in a statistical sense [1]. For large Reynolds numbers these fluctuations are supposed to exhibit universal behavior on scales smaller than the integral one. The elucidation of these properties apparently has to be based on applications of the tools of statistical mechanics. The quantity of main interest is the longitudinal velocity fluctuations $v_i$ on different length scales $L_i$,

$$v_i = u(x + L_i/2, y, z) - u(x - L_i/2, y, z), \quad (1)$$

where $u(x, y, z)$ is the $x$ component of the velocity field at space point $x, y, z$. Based on the idea of an energy cascade, as a fundamental process governing the turbulence, we know from the pioneering works of the 1940s, cf. [1], that the velocity fluctuations are of the order $v_i \sim (\epsilon L_i)^{1/3}$. $\epsilon$ denotes the energy dissipation (transfer) rate. However, it is commonly believed that intermittent fluctuations of the energy dissipation rate alters the scaling behavior. Intermittency effects show up in the changing shape of the probability density functions (pdf) $P_{L_i}(v_i)$ as a function of $L_i$ and consequently lead for the scaling of the moments $\langle (v_i)^n \rangle \sim L_i^{\zeta_n}$ to nonlinear $n$ dependence of the scaling indices $\zeta_n$.

In the present Letter we report on our recent approaches to analyze statistical properties of turbulent cascades. We have started to evaluate conditional probability functions for the velocities $v_i$ of different length scales $L_i$ for a data set consisting of $10^7$ points measured in the center of a free jet with $R_\lambda = 600$ by means of hot-wire anemometry [2]. It is traditional to evaluate the velocity increments (1) by single-probe measurements invoking Taylor's hypothesis of frozen turbulence [1]. Denoting by $\overline{U}$ the mean velocity of the jet, which is much larger than the velocity increments, the length scale $L_i$ is related to a time difference $T_i$ according to

$$L_i = \overline{U} T_i \, . \quad (2)$$

The velocity increment (1) is then estimated by the difference of the velocity signal measured by a single probe at times $t$ and $t + T_i$. It is a well-established fact that in the center of a free jet the small scale structures display local isotropy and homogeneity and that the Taylor hypothesis can be applied. For a more detailed discussion of the experimental setup we refer the reader to [3]. In order to make contact with the theories which are formulated for the spatial velocity increments (1) we have decided to discuss our results in terms of the length scales $L_i$. Using the transformation (2) our results can also be formulated in terms of time scales $T_i$ and thus become independent of the Taylor hypothesis.

A scaling region, indicated by a linear behavior of the third moment $\langle v_i^3 \rangle \approx L_i$, develops in the range between 30 and 200, where the Kolmogorov scale $\eta$ corresponds to $L_i = 0.66$. For obvious reasons we shall use properly scaled velocity variables. From the inertial range we single out an arbitrarily chosen length scale $L_{\rm ref}$ (here $L_{\rm ref} = 324$) and define the scaled velocity increments $\tilde{v}_i = v_i/(L_i/L_{\rm ref})^{1/3}$. If the turbulence obeys the scaling behavior suggested by Kolmogorov in 1941, cf. [1], the statistics of the velocity fields $\tilde{v}_i$ become independent on $L_i$ in the inertial range. Furthermore, without loss of generality we introduce a logarithmic length scale $\lambda_i = \ln(L_{\rm ref}/L_i)$. Note that $\lambda_i$ varies from zero to infinity as $L_i$ decreases from $L_0$ to $\eta$.

It is an important question whether the conditional probability functions fulfill a Chapman-Kolmogorov equation

$$p(\tilde{v}_2, \lambda_2 \mid \tilde{v}_1, \lambda_1) = \int d\tilde{v}_3 \, p(\tilde{v}_2, \lambda_2 \mid \tilde{v}_3, \lambda_3) \times p(\tilde{v}_3, \lambda_3 \mid \tilde{v}_1, \lambda_1) \, , \quad (3)$$

  

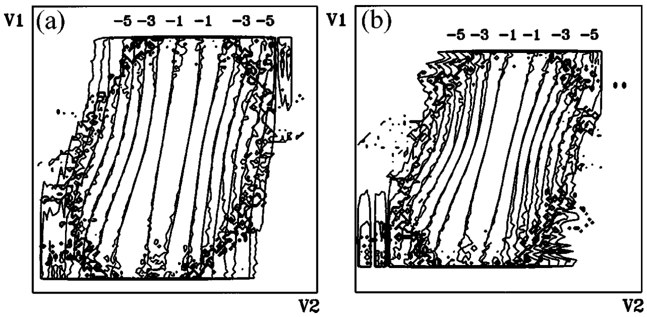


FIG. 1. Verification of the Chapman-Kolmogorov equation. Contour plots ($-50 \leq v_i \leq 50$, $i = 1,2$) of the conditional probability distributions $p(v_2, L_2 \,|\, v_1, L_1)$ in comparison with $p_{\mathrm{cal}}(v_2, L_2 \,|\, v_1, L_1) = \int dv_3\, p(v_2, L_2 \,|\, v_3, L_3) p(v_3, L_3 \,|\, v_1, L_1)$. The contour lines are shown in logarithmic scale. The numbers indicate the contour lines for the values $e^{-n}$, $n = 1, 3, 5$. (a) $L_1 = 224$, $L_3 = 124$, $L_2 = 54$. (b) $L_1 = 124$, $L_3 = 54$, $L_2 = 34$; note that $L_2 = 34$ is already out of the inertial range. A velocity value of 50 correponds to a measured velocity of 13, expressed in units of local Reynolds number at the detector [3].

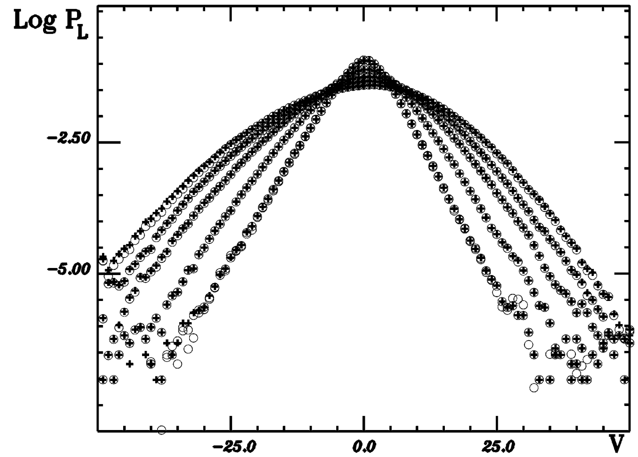


FIG. 2. Comparison of probability density functions for $L_i = 24, 54, 124, 224, 424$ obtained directly from the data set ($\circ$), and evaluated ($+$) by iterating the Chapman-Kolmogorov equation starting form from $P_{L_1}(v_1)$ for $L_1 = 1024$ using the experimentally determined conditional probabilities $p(v_j, L_j \,|\, v_i, L_i)$.

where $\lambda_1 < \lambda_3 < \lambda_2$. This equation is a necessary condition for the statistics of the turbulent cascade to be Markovian. It means that the transition probabilities from scale $\lambda_1$ to $\lambda_2$ can be subdivided into a transition from $\lambda_1$ to $\lambda_3$ and then from $\lambda_3$ to $\lambda_2$. It is related with the idea [1] that the turbulent cascade is generated by a local transfer mechanism.

In the Markovian case the conditional probabilities $p(\tilde{v}_i, \lambda_i \,|\, \tilde{v}_j, \lambda_j)$ determine all the $n$-point probability distributions $p(\tilde{v}_n, \lambda_n; \ldots, \tilde{v}_i, \lambda_i; \ldots, \tilde{v}_1, \lambda_1)$, $n = 1, ..., \infty$. We have checked the validity of the Chapman-Kolmogorov equation for different $\lambda_i$ triplets. Figure 1 compares conditional probability distributions $p(\tilde{v}_2, \lambda_2 \,|\tilde{v}_1, \lambda_1)$ with the ones calculated ($p_{\mathrm{cal}}$) according to (3). There only are visible deviations resulting from a finite resolution of the statistics in the outer regions. We want to point out that our results indicate that Eq. (3) holds in the whole range between $L_0$ and $\eta$. Figure 2 shows that we are able to obtain all $P_{L_n}(v_n)$ by iterating the Chapman-Kolmogorov equation starting from a pdf at large scale with remarkable precision.

The Chapman-Kolmogorov equation can be formulated in differential form [4] leading to an evolution equation for the conditional probability distribution $p(\tilde{v}_2, \lambda_2 \,|\tilde{v}_1, \lambda_1)$. If the Kramers-Moyal coefficients $\tilde{D}^n(\tilde{v}_2, \lambda_2)$, defined as

$$\tilde{D}^n(\tilde{v}_2, \lambda_2) = \frac{1}{n!} \lim_{\lambda_3 \to \lambda_2} \frac{1}{\lambda_3 - \lambda_2} \int d\tilde{v}_3 (\tilde{v}_3 - \tilde{v}_2)^n \times p(\tilde{v}_3, \lambda_3 \,|\, \tilde{v}_2, \lambda_2)\,, \tag{4}$$

all vanish for $n \geq 3$ we are led to a Fokker-Planck equation,

$$\frac{\partial}{\partial \lambda_2} p(\tilde{v}_2, \lambda_2 \,|\, \tilde{v}_1, \lambda_1) = \left[ -\left(\frac{\partial}{\partial \tilde{v}_2}\right) \tilde{D}^1(\tilde{v}_2, \lambda_2) + \left(\frac{\partial}{\partial \tilde{v}_2}\right)^2 \tilde{D}^2(\tilde{v}_2, \lambda_2) \right] \times p(\tilde{v}_2, \lambda_2 \,|\tilde{v}_1, \lambda_1)\,, \tag{5}$$

and the stochastic process is completely characterized by the drift and diffusion coefficients $\tilde{D}^1(\tilde{v}_i, \lambda_i)$, $\tilde{D}^2(\tilde{v}_i, \lambda_i)$. We have calculated approximations to the limits Eq. (4) defining the first four Kramers-Moyal coefficients and are able to show that the approximants of the third and fourth order coefficients tend to zero whereas the first and second coefficients have well-defined limits [5]. The Pawulas theorem [4] infers that all higher Kramers-Moyal coefficients vanish (provided they exist) if $D^4$ vanishes.

Figure 3 shows the drift and diffusion coefficients calculated for the turbulent cascade from the experimental data for various values of $\lambda_i$ or $L_i$, respectively. It seems that the drift coefficient becomes linear in $\tilde{v}_i$ and independent

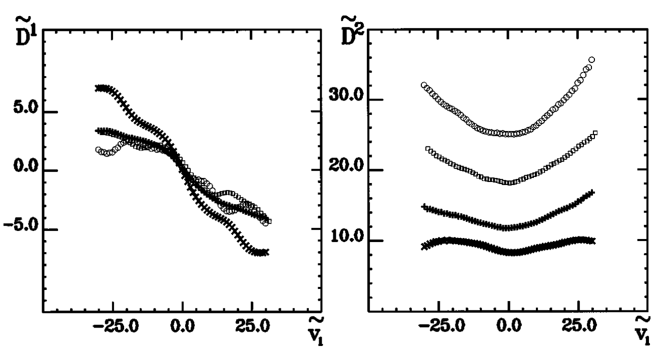


FIG. 3. Scaled drift and diffusion coefficients $\tilde{D}^1(\tilde{v}_1, \lambda_1)$, $\tilde{D}^2(\tilde{v}_1, \lambda_1)$. [$L = 20$ ($\times$), 54 ($+$), 124 ($\square$), 224 ($\circ$)].

of $\lambda_i$ in the inertial range. The drift coefficient for $L_i = 20$ differs since it is not contained in the scaling region. Apparently, the diffusion coefficient is a function of $\tilde{v}_i$ and depends on $L_i$ also in the scaling region.

Let us draw some conclusions from our results. It should be kept in mind that we have not yet proven that the $n$-point probability distributions form a Markov chain. However, non-Markovian processes which fulfill the Chapman-Kolmogorov equation should be rather rare. If we assume that turbulence is actually a Markovian process in space the following expression [4] contains the complete information on the statistics:

$$p(\tilde{v}_n, \lambda_n; \ldots, \tilde{v}_1, \lambda_1) = \frac{1}{\sqrt{\prod_{j=1}^{n-1} 4\pi \tilde{D}^2(\tilde{v}_j, \lambda_j) d\lambda}} \exp\left\{ -\sum_{j=1}^{n-1} d\lambda \frac{[(\tilde{v}_{j+1} - \tilde{v}_j)/d\lambda - \tilde{D}^1(\tilde{v}_j, \lambda_j)]^2}{4\tilde{D}^2(\tilde{v}_j, \lambda_j)} \right\} P_{\lambda_1}(\tilde{v}_1, \lambda_1), \quad (6)$$

where $\lambda_j = j d\lambda$. In the limit $d\lambda \to 0$ this is a path integral representation of a probability functional $p(\tilde{v}(\lambda))$, where $\tilde{v}(\lambda)$ is now a function of the continuous variable $\lambda$.

It is well known [4] that the Fokker-Planck equation is equivalent to a stochastic evolution equation,

$$\tilde{v}(\lambda + d\lambda) = \tilde{v}(\lambda) + d\lambda \tilde{D}^1(\tilde{v}, \lambda) \sqrt{\tilde{D}^2(\tilde{v}, \lambda) d\lambda}\, \eta_\lambda, \quad (7)$$

where $\eta_\lambda$ is a normally distributed random variable. This equation leads us to the following interpretation. The stochastic evolution equation yields a realization of a turbulent velocity field $\tilde{v}(\lambda)$, where the systematic behavior is governed by the drift term, i.e., a classical path, whereas the diffusion term contains fluctuations around this path. This underlines the importance for an investigation of these coefficients for different types of turbulent flows.

Let us consider a scenario where drift and diffusion coefficients become independent on the length scale $\lambda$. This may happen for the limiting case of infinite Reynolds numbers. Then the distribution [6]

$$P_{\mathrm{stat}}(\tilde{v}_i) = \frac{N}{\tilde{D}^2(\tilde{v}_i)} \exp\left[ \int_{\tilde{v}_0}^{\tilde{v}_i} dv' \tilde{D}^1(v')/\tilde{D}^2(v') \right] \quad (8)$$

is a stationary solution of the Fokker-Planck equation in the inertial range. Although the shape of this distribution may differ considerably from a Gaussian no intermittency connected with a change of the shape of the distribution shows up provided that this distribution is actually established in the experiment at large scales. Consequently no deformation of the shape of the probability distribution will take place under the development of the $\lambda$-independent Fokker-Planck equation. The scaling behavior suggested by Kolmogorov in 1941 is valid.

Since in each experiment the turbulent flow is driven at the integral length scale $L_0$ the distribution for $L < L_0$ has to match a distribution at $L_0$ which may be different from the stationary one. In that case one has to take into account nonstationary distributions of the Fokker-Planck equation leading to intermittency effects.

Let us consider the following scenario which will lead us to the scaling behavior of the velocity increments (1) suggested by Kolmogorov and Oboukhov [7] in 1962 (K 62 scaling). Taking into account fluctuations of the local energy dissipation rate and assuming a log-normal distribution for this quantity they concluded that the $n$th order moment $v_i^n$ should scale according to

$$\langle (v_i)^n \rangle \approx L_i^{n/3 - \mu n(n-3)/18}, \quad (9)$$

where $\mu$ is the Kolmogorov intermittency coefficient. From Fig. 3 it is evident that the diffusion coefficient contains, besides a $\tilde{v}$ independent part, also a contribution quadratic in $\tilde{v}$. Thus we are guided to consider the hypothetical case where the diffusion coefficient becomes purely quadratic in $\tilde{v}$ and the drift term linear, $\tilde{D}^1 = -\gamma \tilde{v}$, $\tilde{D}^2 = Q\tilde{v}^2$. This corresponds to a multiplicative noise term in the Langevin equation (7). In this case now a stationary solution of the Fokker-Planck equation is the distribution $p_{\mathrm{stat}}(v_i) = \delta(v_i)$. As indicated above, one has to resort to nonstationary solutions. In fact, the Fokker-Planck equation (5) for the present case yields the following expression for the $n$th order moments [8]

$$\begin{aligned} \langle \tilde{v}^n \rangle (\lambda) &= -\langle \tilde{v}^n \rangle (0) \exp\{[-\gamma n + Qn(n-1)]\lambda\} \\ &= c(L_0/L)^{-\gamma n + Qn(n-1)}. \end{aligned} \quad (10)$$

Using Kolmogorov's $-4/5$ law [1] that the third order moment $\langle \tilde{v}^3 \rangle$ of the scaled velocity increment $\tilde{v}$ should be constant in the inertial range we obtain $\gamma = 2Q$. The Kolmogorov intermittency exponent is related to the constant $Q$ of the diffusion coefficient $Q = \mu/18$. Note that with the definition of $\tilde{v}$, $\langle v^n \rangle \approx \langle \tilde{v}^n \rangle r^{n/3}$.

To conclude, by a detailed analysis of experimental data of a fluid dynamical experiment we were able to obtain a phenomenological description of the statistical properties of a turbulent cascade using a Fokker-Planck equation. As an evolution equation for the probability density function of the velocity increment (1) this equation contains the information on the changing shape of the distribution as a function of the scale $L$. Thus information on the observed intermittency of the turbulent cascade is obtained. Based on simplified assumptions on the drift and diffusion coefficient we have discussed two scenarios in order to indicate that both, the Kolmogorov 41 and 62 scalings, are contained as possible behavior in the present description. However, the experimentally determined drift and diffusion coefficients are more complicated, at least for the Reynolds number investigated here. The drift term may well be linear in $\tilde{v}$ and $\lambda$ independent in the inertial range, but the diffusion coefficient appears to depend on $\lambda$ as well as on $\tilde{v}$. Therefore, it seems to be highly important to investigate these functions for flows with different Reynolds numbers and different experimental setups.

We acknowledge discussions with P. Talkner, K. Zeile, H. Brand, S. Grossmann, and A. Reeh. J. P. acknowledges the cooperation with B. Castaing and B. Hebral. He also acknowledges the financial support of the Deutsche Forschungsgemeinschaft.

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1975); U. Frisch, *Turbulence* (Cambridge University, Cambridge, England, (1995).

[2] J. Peinke, R. Friedrich, F. Chilla, B. Chabaud, and A. Naert, Z. Phys. B **101**, 157 (1996).

[3] B. Chabaud, A. Naert, J. Peinke, F. Chilla, B. Castaing, and B. Hebral, Phys. Rev. Lett. **73**, 3227 (1994).

[4] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984); P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).

[5] R. Friedrich and J. Peinke, "Statistical Properties of a Turbulent Cascade," Physica D (to be published).

[6] A formally similar formula has been proposed for passive scalar in turbulence by S. B. Pope and E. S. C. Ching, Phys. Fluids A **5**, 1529 (1993), but with different definitions of $D^1$ and $D^2$.

[7] A. M. Oboukhov, J. Fluid Mech. **13**, 77 (1962); A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).

[8] Here we see that the introduction of the logarithmic length scale $\lambda$ is linked to scaling behavior.