

Multiscale approach to inhomogeneous cosmologies

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Abstract

The backreaction of inhomogeneities onto the global expansion of the universe suggests a possible link of the formation of structures to the recent accelerated expansion. In this paper, the origin of this conjecture is illustrated and a model that allows a more explicit investigation is discussed. Additionally to this conceptually interesting feature, the model leads to a Λ CDM like distance-redshift relation that is consistent with supernova data.

Averaged Equations For The Expansion Of The Universe

The averaging problem

The evolution of our universe is described by Einstein's equations of General Relativity. These are ten coupled differential equations for the coefficients of the metric that describes our spacetime. In the cosmological case, where we are only interested in the overall evolution and not in the detailed local form of the inhomogeneous metric, cosmologists widely work with the assumption that the global evolution is described by the single homogeneous and isotropic solution of the Einstein Equations: They use a homogeneous isotropic fluid as the content of the universe that sources the evolution of the homogeneous and isotropic Robertson Walker (RW) metric and therefore determines the lapse of the expansion. This latter is thereby condensed into the evolution of a single quantity, the scale factor $a(t)$. The fundamental question, dating back to Shirokov and Fisher (1963) and most prominently raised by George Ellis in 1983 (Ellis (1983)), is then, if this procedure leads to the correct description of the global behaviour of our spacetime. Are the Einstein equations the correct effective equations that describe the average evolution, even if they are local equations?

To address this question, one has to find a way to explicitly average the equations. This is necessary, because if one performs an average of an inhomogeneous metric whose time evolution has been determined by the use of the ten Einstein equations, one finds a result that differs from the classical case above. This is already evident from the fact that Einsteins equations are nonlinear. But already at the linear level, as soon as the volume element of the domain of averaging is time dependent, this non commutation is present. This is easy to see from a derivation of the definition of the average of a scalar quantity f , $\langle f \rangle_D(t) := \int_D f(t, X) \sqrt{^{(3)}g(t, X)} d^3 X / V_D(t)$ with respect to time which provides $\partial_t \langle f \rangle_D - \langle \partial_t f \rangle_D = \langle f \theta \rangle_D - \langle f \rangle_D \langle \theta \rangle_D \neq 0$, θ being the local expansion rate. So, if we take an implicitly averaged metric, i.e. the RW metric, and the effective homogeneous matter source, and then calculate its time evolution with the standard Friedmann equation, this will not give the average of the time evolved quantity. Or to put it short, time evolution and averaging do not commute, often strikingly written as $G_{\mu\nu}(\langle g_{\mu\nu} \rangle) \neq \langle G_{\mu\nu}(g_{\mu\nu}) \rangle$ and depicted in Fig. 1.

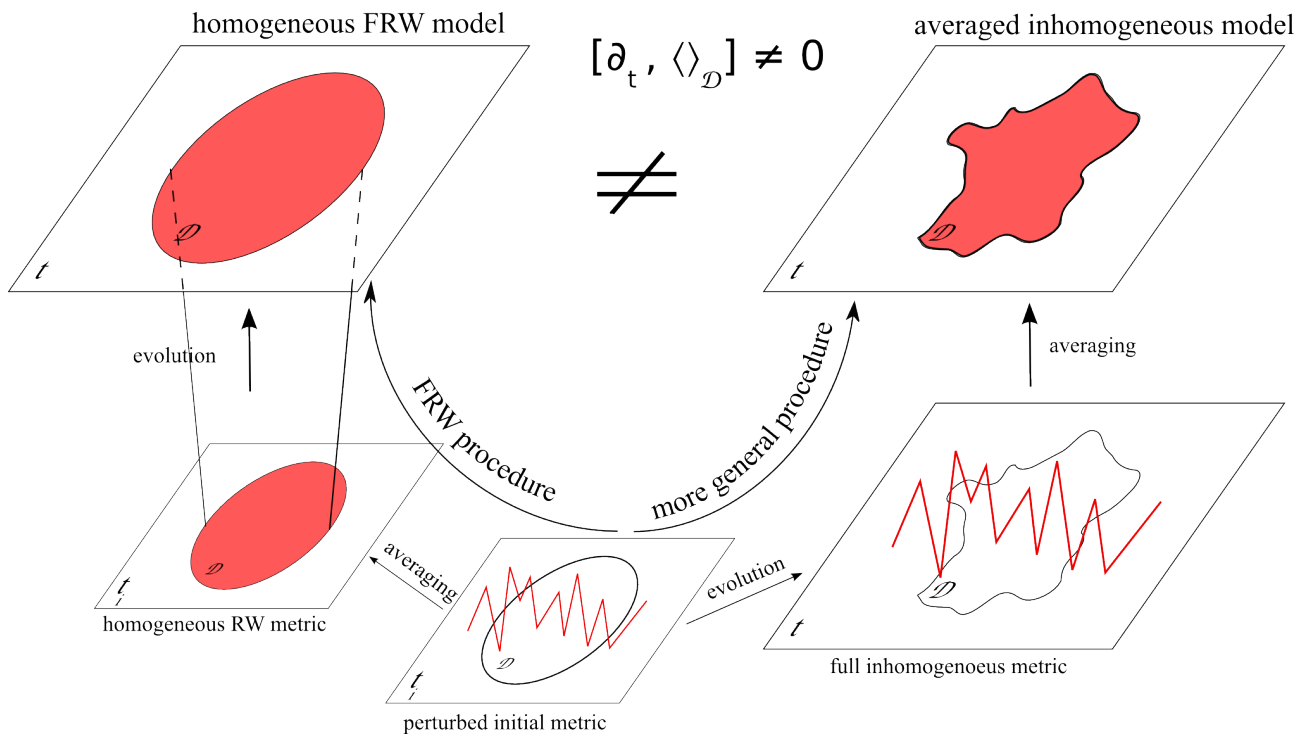


Figure 1: Illustration of the cause of the departure of the average evolution from the standard Friedmann solutions: The noncommutativity of spatial averaging and time evolution. The standard Friedmannian picture is the left branch, where you first average the inhomogeneous matter distribution and calculate the evolution of this homogeneous soup by the Einstein equations. In contrast to that, in the averaged model shown by the right branch, the perturbation evolution is still performed with the full metric and only then, the average is taken. The two approaches give different results. But the main question remains: How big is this difference?

Provenance of the averaged equations

In the recent literature, the question of how big the difference between these two approaches is, has received growing interest, mainly due to attempts to relate it to the dark energy problem (Räsänen (2004), Kolb *et al.* (2005)). Since then, there have been many calculations trying to quantify the impact of this noncommutativity of time evolution and averaging. The direct way of trying to average the Einstein equations in their tensorial form turned out to be very difficult, because it is not clear how to define a meaningful average of tensors. Therefore the most popular scheme to work with is still the one by Buchert (Buchert (2000) and Buchert (2001)). In this approach, one uses the ADM equations to perform a 3+1 split of the spacetime into spatial hypersurfaces orthogonal to the fluid flow. The source is also in this case often taken to be a perfect fluid and the equations take their simplest form in the frame of an observer comoving with the fluid. After this split one can identify scalar, vector and tensor parts of the resulting equations. For the scalar sector there is then a straightforward definition of an average quantity as the integral of the scalar function over a comoving domain of the spatial hypersurface, divided by the volume of this domain. The use of this definition astonishingly provides a set of two differential equations for the average scale factor of the averaging domain, that resembles closely the Friedmann equations in the homogeneous case:

$$3 H_D^2 = 8\pi G \langle \rho \rangle_D - \frac{1}{2} \langle R \rangle_D - \frac{1}{2} Q_D + \Lambda \quad (1)$$

$$\frac{\ddot{a}_D}{a_D} = -4\pi G \langle \rho \rangle_D + Q_D + \Lambda \quad (2)$$

This is surprising, because in this approach it is not necessary to constrain the matter source to a completely homogeneous one, but one can have arbitrarily large spatial variations in the density. There are, however, two important differences between the general averaged evolution equations (1) and (2) for the volume scale factor and the Friedmann equations. First of all, there is one extra term Q_D , which is called the kinematical backreaction term. It encodes the departure of the matter distribution in the spatial hypersurface from a homogeneous distribution. This is because it is defined as the variance of the local expansion rate of the spacetime minus the variance of the shear inside the domain \mathcal{D} . Therefore, if the expansion fluctuations are bigger than the shear fluctuations, Q_D is positive and contributes to the acceleration of the spatial domain (c.f. Eq. (2)). For dominating shear fluctuations, Q_D is negative, and decelerates the domain's expansion. This effective term Q_D that emerges from the explicit averaging procedure, induces the second difference to the standard Friedmann equations. By the integrability condition of the equations for the average scale factor (Eqs. (1) and (2)), Q_D is connected to the average internal curvature of the domain \mathcal{D} . Unlike in the

Friedmann case, where the curvature scales as a_D^{-2} , the dependence of the average curvature $\langle R \rangle_D$ on a_D , has not necessarily the form of a simple power law. In fact it can be shown that the curvature picks up an integrated contribution of the variation of Q_D and evolves in this way generically away from the flat initial conditions, expected to emerge from inflation.

Uncommon properties of the averaged model

These two changes to the standard Friedmann equations may alter the expansion history considerably. Regarding Eq. (2), it is easy to see, that for $Q_D > 4\pi G \langle \rho \rangle_D$ there may even be an accelerated epoch of expansion without the presence of a cosmological constant. It may seem surprising that even in a universe only filled with a perfect fluid of ordinary (or dark) matter, there may be an effective acceleration of a spatial domain \mathcal{D} . This occurs, because in the calculation

of the average expansion rate of \mathcal{D} , the local expansion is weighted with its corresponding volume. Therefore, faster expanding subregions of \mathcal{D} , which will by their faster growth occupy a larger and larger volume fraction of \mathcal{D} , will

eventually dominate its expansion. Subregions that slow down their expansion, will finally only occupy a negligible fraction of the volume of \mathcal{D} . This means that a volume weighted average of the expansion rate will start with a value between the one of the slow and fast expanding regions, when they have still a comparable size, but will be driven towards the value of the fastest expanding domain in the late time limit. This growth in the average expansion rate corresponds to an acceleration of the growth of the volume scale factor.

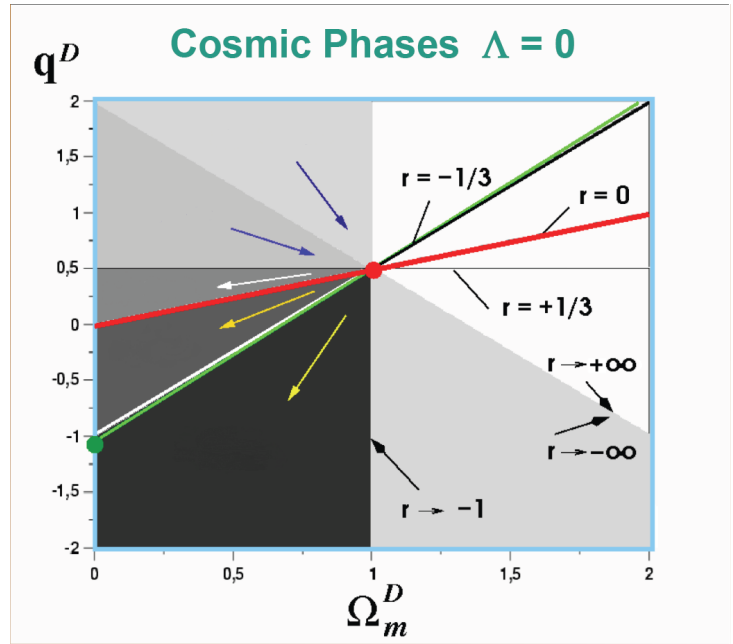


Figure 2: Phase space of the solutions to the averaged equations Eqs. (1) and (2). The deceleration parameter $q^D = 1/2 \Omega_m^D + 2 \Omega_Q^D$ is in the present case of $\Lambda = 0$ effectively a measure of Ω_Q^D and Ω_m^D is the matter parameter on \mathcal{D} . The point in the middle is the EdS model and the line $r=0$ encompasses the Friedmann models i.e. $\Omega_Q^D = 0$.

A convenient way to illustrate the possible departure of the solutions of the equations for the average scale factor, from the Friedmann solution, is the phase space diagram in Fig. 2. For a universe without cosmological constant every path in this two dimensional plane, corresponds to a solution of the averaged equations. Scaling solutions for which the a_D -dependence of the kinematical backreaction \mathcal{Q}_D is given by a single power law a_D^n , show up as straight lines. The Friedmann solutions lie on the line $r=0$. A first phase space analysis of this parameter space in Buchert *et al.* (2006) has shown, that the EdS model in the middle is an unstable saddle point. Perturbations of the homogeneous state in the matter dominated era, will therefore drive the universe away from it in the direction indicated by the arrows. This is also the region in which the \mathcal{Q}_D term is positive. Therefore the instability of the Friedmann model leads naturally to accelerated expansion, if the phase space is traversed rapidly enough. A more elaborated phase space analysis may soon be found in Roy and Buchert (2011).

The important changes to the Friedmann model that emerge when passing to explicit averages, may be summarized in the following generalized concepts:

1. The single homogeneous and isotropic solution of Einstein's equations is replaced by explicit averages of the equations of general relativity.
2. The background is now generically interacting with the structure in the spatial hypersurfaces.
3. The inhomogeneities do no longer average out on the background.
4. The full Riemannian curvature degree of freedom is restored and the equations no longer are restricted to a constant curvature space.

A more detailed review of our current understanding of the description of the average evolution may be found in Buchert (2008) and Räsänen (2006).

Partitioning Models

To build a specific model using the above framework there have been several attempts. As the equations for $a_D(t)$ are not closed, one has to impose, like in the Friedmann case, an equation of state for the fluid. The problem is here, that the fluid composed of backreaction \mathcal{Q}_D and average curvature $\langle R \rangle_D$ is only an effective one. Therefore it is not clear which equation of state one should choose. In Roy and Buchert (2009) for example, the equation of state of a Chaplygin gas has

been used. Another approach in Buchert *et al.* (2006) has been to take a constant equation of state which leads to simple scaling solutions for the a_D -dependence of \mathcal{Q}_D and $\langle R \rangle_D$. Those will be generalized here by the partitioning model, described in detail in Wiegand and Buchert (2010).

Model construction

As the name suggests, this is done by a partitioning of the background domain \mathcal{D} into subdomains. To have a physical intuition about the evolution of the subdomains, a reasonable choice is to partition \mathcal{D} into overdense \mathcal{M} - and underdense \mathcal{E} -regions. \mathcal{M} and \mathcal{E} regions will also obey the average equations (1) and (2) and there are consistency conditions, resulting from the split of the \mathcal{D} equations into \mathcal{M} and \mathcal{E} equations, that link the evolution of \mathcal{D} , \mathcal{M} and \mathcal{E} . The reason for the split is that it offers the possibility to replace the unintuitive backreaction parameter \mathcal{Q}_D by a quantity that illustrates more directly the departure from homogeneity, namely the volume fraction of the overdense regions $\lambda_{\mathcal{M}}$. As explained above, $\lambda_{\mathcal{M}}$ is expected to decrease during the evolution and it is this decrease that drives the acceleration. The physical motivation to split into over and underdense regions is, that from the structure of the cosmic web, one may expect \mathcal{E} regions, which are mainly composed of voids, to be more spherical than \mathcal{M} regions. On \mathcal{E} , the expansion fluctuations should therefore be larger than the shear fluctuations and so $\mathcal{Q}_{\mathcal{E}}$ should be positive. The shear fluctuation dominated \mathcal{M} regions should have a negative $\mathcal{Q}_{\mathcal{M}}$. This increases the difference between the faster expanding \mathcal{E} and the decelerating \mathcal{M} regions even further and therefore magnifies the expansion fluctuations on \mathcal{D} that drive acceleration via a positive \mathcal{Q}_D . The fact that $\mathcal{Q}_{\mathcal{M}}$ and $\mathcal{Q}_{\mathcal{E}}$ are nonzero is the main difference to a similar model of Wiltshire (Wiltshire (2007a), Wiltshire (2007b)).

The generalization of the single scaling laws mentioned above, is achieved by imposing the scaling on \mathcal{M} and on \mathcal{E} . In Li and Schwarz (2008) the authors showed that \mathcal{Q}_D and $\langle R \rangle_D$ may be expressed in a Laurent series starting at a_D^{-1} and a_D^{-2} respectively. This behaviour breaks down if the fluctuations with respect to the mean density become of order one. This happens later on \mathcal{M} and on \mathcal{E} because on these regions the mean values lie above, resp. below, the global mean and

therefore the fluctuations on \mathcal{M} with respect to this overdense mean are smaller than the variation between the peaks on \mathcal{M} and the troughs on \mathcal{E} . Therefore the a_D^{-1} scaling for \mathcal{Q}_D on \mathcal{M} and \mathcal{E} is expected to hold true even if the \mathcal{D} regions already depart from this perturbatively determined behaviour. The partitioning model is therefore the first step of a generalization to an arbitrary nonlinear behaviour of \mathcal{Q}_D and $\langle R \rangle_D$ on the global domain \mathcal{D} .

Using this Ansatz for the \mathcal{Q}_D -evolution on \mathcal{M} and \mathcal{E} , and exploiting the consistency conditions for the partitioning, it is possible to arrive at a model that depends only on three parameters: The Hubble rate today H_{D_0} , the matter density today $\Omega_m^{D_0}$ and the volume fraction of the overdense regions today $\lambda_{\mathcal{M}_0}$. In this parametrization H_{D_0} sets only the time scale, so we may fix the evolution with $\Omega_m^{D_0}$ and $\lambda_{\mathcal{M}_0}$. Assuming for $\Omega_m^{D_0}$ the concordance value of 0.27, the model shows that the more structure there is, indicated by a low value of $\lambda_{\mathcal{M}_0}$, the higher the acceleration on the domain \mathcal{D} will be.

To analyze what the order of magnitude of $\lambda_{\mathcal{M}}$ is today, an N-body simulation was studied. Smoothing the point distribution on $5h^{-1}\text{Mpc}$ and applying a simple number count for the determination of $\lambda_{\mathcal{M}_0}$, a value of 0.09 was obtained. An analysis by a Voronoi tessellation gave about 0.02. To obtain a definite value, this analysis clearly has to be improved. One would have to use a proper SPH smoothing and trace the overdense regions, that are fixed in the initial conditions, until today. But in any case, $\lambda_{\mathcal{M}_0}$ seems to be in the region below 0.1. Interestingly

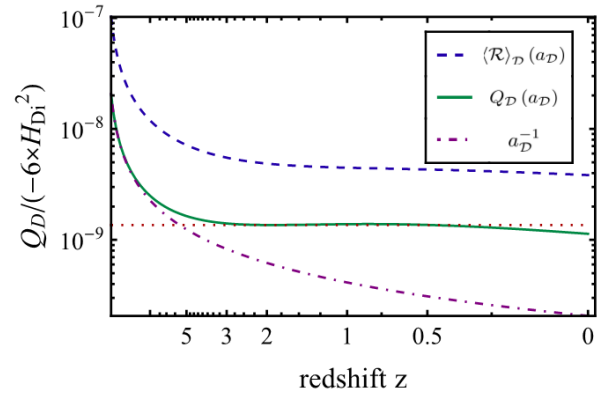


Figure 3: Evolution of the backreaction parameter \mathcal{Q}_D in course of cosmic time. From a value of Ω_Q^D of around 10^{-7} at a redshift of $z \approx 1000$ \mathcal{Q}_D first decreases like a_D^{-1} and becomes approximately constant for a long period of the evolution when $\lambda_{\mathcal{M}_0}$ gets important.

enough, this is the region for $\lambda_{\mathcal{M}_0}$ which leads to a nearly constant \mathcal{Q}_D on \mathcal{D} , shown in Fig. 3. So for low values of $\lambda_{\mathcal{M}_0}$, \mathcal{Q}_D acts like a cosmological constant.

Observational strategies

This may also be seen by a fit of the model to luminosity distances of supernovae (SN). To convert the evolution of the average scale factor a_D into luminosity distances, we use the result of Räsänen (2009), who investigated the propagation of light in inhomogeneous universes and provided a formula linking the observed luminosity distance to the volume scale factor a_D . The resulting probability contours in the parameter space $\Omega_m^{D_0}$ - $\lambda_{\mathcal{M}_0}$ in Fig. 4 show, that indeed the region around an $\Omega_m^{D_0}$ of 0.3 and a $\lambda_{\mathcal{M}_0}$ below 0.1 is favoured by the data.

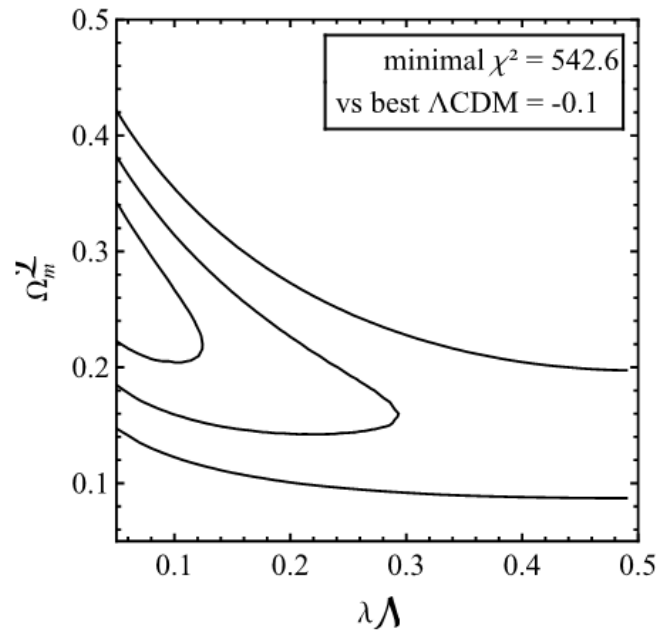


Figure 4: Probability contours of a fit of the partitioning model to the Union2 SN data set. The model gives a comparably reasonable fit as a simple Λ CDM model.

To further test the viability of the model, it will soon be compared with more observational data. But as it is probably possible to fit also this, as indicated by the success of Larena *et al.* (2009), we need a quantity that will definitively allow to find out whether this model or one with a real cosmological constant fits better. To decide that, one may use a quantity introduced by Clarkson *et al.* (2008), namely the C-function

$$C(z) = 1 + H^2(D D'' - D'^2) + H H' D D'$$

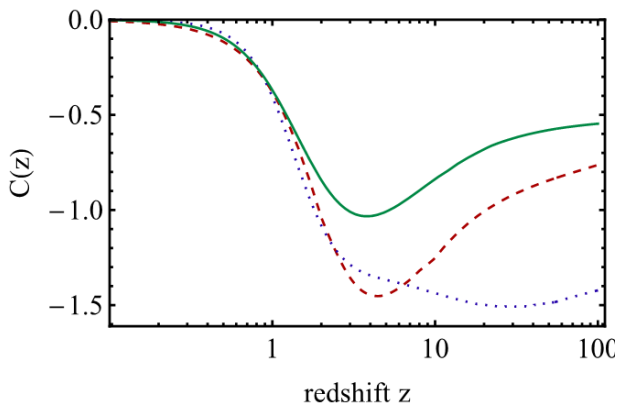


Figure 5: Plot of Clarkson's C-function for several models discussed in Wiegand (2010). The model presented here is shown as the dotted line. The dashed line is its nonperturbative generalization and the solid line is a single scaling model. For every Friedmann model $C(z)=0$ which allows one to distinguish averaged models from FRW models.

et al. (2008), namely the C-function

It consists of derivatives of the Hubble rate $H(z)$ and the angular diameter distance $D(z)$, and is constructed such that for every FRW model it is exactly 0 for any z . For the partitioning model this is generically not the case and for several choices of parameters the difference is shown in Fig. 5. Unfortunately, the function $C(z)$ is too complicated to evaluate it using present day data, but as shown in Larena *et al.* (2009), Euclid may be able to derive its values.

Conclusion

Routing the accelerated expansion back to inhomogeneities would be an interesting possibility to avoid problems with a cosmological constant, such as the coincidence problem and would give the sources of acceleration a physical meaning. Perturbatively analyzed it is clear that the effect is way too small to give rise to an acceleration on the scale of the Hubble volume (Brown *et al.* (2009a), Brown *et al.* (2009b)). There are however semirealistic nonperturbative models like Räsänen (2008) that show that a considerable effect is not excluded. Furthermore it has been shown why perturbative models are not able to give a definite answer on the magnitude of the effect Räsänen (2010). So it seems that the question of how the growth of structure influences the overall expansion of the universe is still an open issue. The presented partitioning model has shown, that a Λ CDM like expansion is possible in the context of these models, without the prior assumption of $Q_D = \text{const.}$, which emerges here more naturally. Furthermore it has been shown how this is related to the formation of structure described by the parameter λ_M . Finally, using the Clarkson's C-function, we will have, at the latest with the data from the Euclid satellite, a handle on how to distinguish acceleration due to inhomogeneities from the presence of a strange fluid. All in all, these are promising prospects for the future...

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