Statistical Investigation of Turbulent Mixing by Means of Turbulent Line Segments

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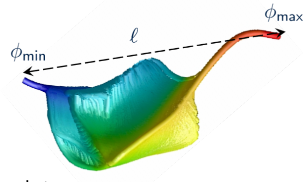
Motivation – Two-Point Statistics in Turbulent Flows

- Turbulent flows are **non-linear** with a strong interaction between **various scales**
- Description by two-point statistics

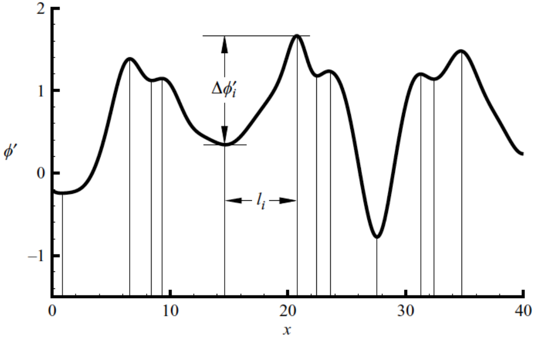
1. **Kolmogorov/Yaglom theory:**
   Define velocity or scalar increment: $\Delta\phi = \phi(\boldsymbol{x} + \boldsymbol{r}) - \phi(\boldsymbol{x})$
   Yaglom equation: $-\langle(\Delta u)(\Delta\phi)^2\rangle = \frac{2}{3}\langle\chi\rangle r$
   Komogorov equation: $-\langle(\Delta u)^3\rangle = \frac{4}{5}\langle\varepsilon\rangle r$



2. **Dissipation Elements (Wang and Peters 2006)**
   Decomposition of scalar fields into ensembles of gradient trajectories that share the same minima/maxima points
   Parameterization by individual length ℓ and scalar difference $\Delta\phi$

3. **Present Approach: Linear Line Segments** Decomposition of the turbulent signal along a straight line

Linear Line Segments – Decomposition of the Turbulent Signal



- **Decompose** the turbulent signal along a straight line into **linear segments** between local minimum and maximum points
- Linear length ℓ
- Scalar difference $\Delta\phi$
- Additionally: mean gradient $g = \dfrac{\Delta\phi}{\ell}$

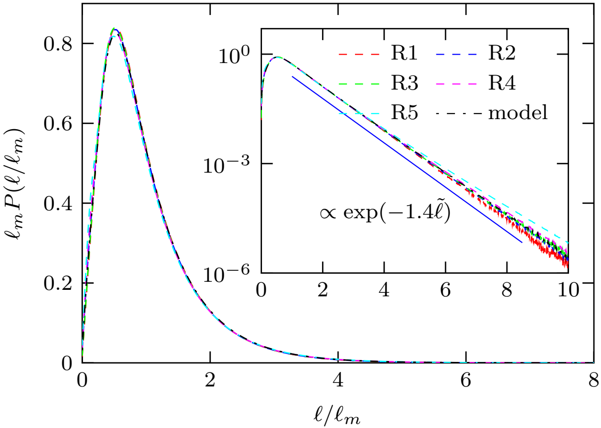
Direct Numerical Simulation of Turbulent Mixing

- Incompressible Navier-Stokes equations + passive scalar with imposed mean gradient
- Pseudo-spectral method in triply periodic box
- DNS conducted on JUQUEEN (IBM BlueGene/Q) with up to 524,288 threads

	R0	R1	R2	R3	R4	R5
N	512	1024	1024	2048	2048	4096
Re_λ	88	119	184	215	331	529
ν	0.01	0.0055	0.0025	0.0019	0.0010	0.00048
$\kappa_{\max}\eta$	3.57	4.54	2.66	4.01	2.30	2.70
$S(\partial_{\parallel}\phi)$	1.65	1.73	1.60	1.55	1.56	1.36
$S(\partial_L u)$	-0.52	-0.54	-0.55	-0.57	-0.59	-0.64
t_{avg}/τ	100	30	30	10	10	2
ensembles	189	62	61	10	10	5

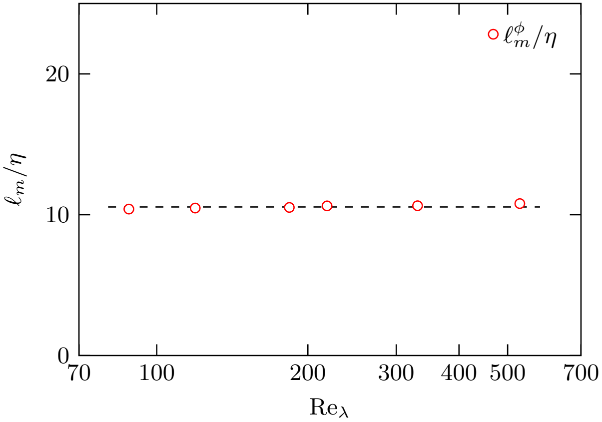
- 45 million core hours computational time for all cases

Normalized Marginal pdf $P(\ell)$



- pdf of normalized length becomes **quasi-universal** when normalized by the mean length ℓ_m.

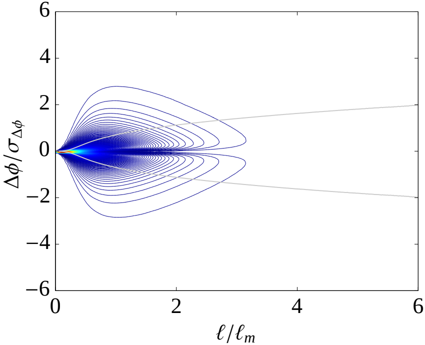
Scaling of the Mean Length ℓ_m with Reynolds Number

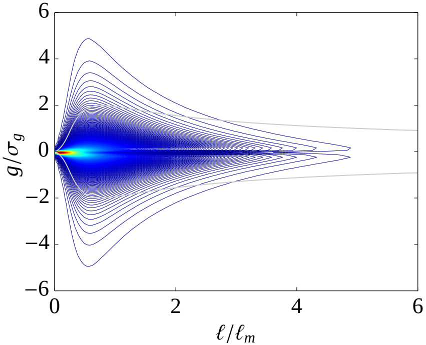


- Scaling of the mean length ℓ_m with the Kolmogorov length $\eta \Rightarrow \boxed{\dfrac{\ell_m}{\eta} \approx 10.5}$

Statistical Description

Joint pdf $P(\Delta\phi, \ell)$ and $P(g, \ell)$ for positive and negative segments





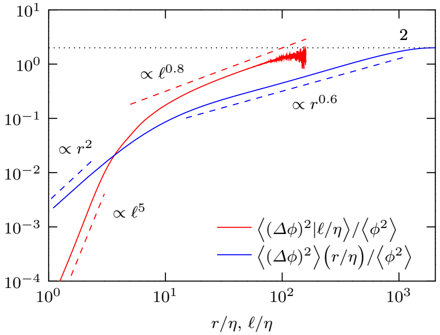
Relation between $P(\Delta\phi, \ell)$ and $P(g, \ell)$:

$$P_{\ell g}(\ell, g) = \int_{-\infty}^{\infty} \delta\left(g - \frac{\Delta\phi}{\ell}\right) P_{\ell\Delta\phi}(\ell, \Delta\phi)\mathrm{d}(\Delta\phi)$$

$$= \int_{-\infty}^{\infty} \ell\delta\left(g\ell - \Delta\phi\right) P_{\ell\Delta\phi}(\ell, \Delta\phi)\mathrm{d}(\Delta\phi) = \ell\, P_{\ell\Delta\phi}(\ell, g\ell)$$

Comparison of Conditional Moments with Classical Structure Functions

- Relation to jpdf: $P(\Delta\phi|\ell) = \dfrac{P(\Delta\phi, \ell)}{P(\ell)}$
- Conditional moments: $\langle(\Delta\phi)^n|\ell\rangle = \int(\Delta\phi)^n P(\Delta\phi|\ell)\mathrm{d}(\Delta\phi)$



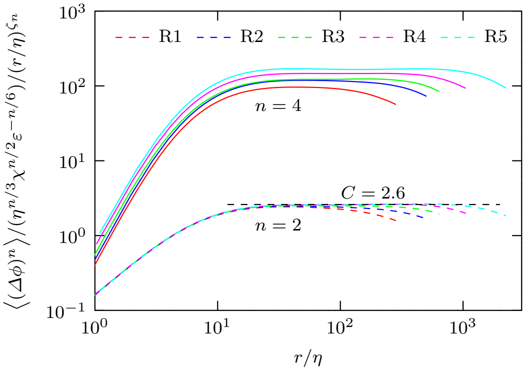
- Conventional structure function: $\langle(\Delta\phi)^2\rangle(\boldsymbol{r}) = \langle(\phi(\boldsymbol{x}+\boldsymbol{r}) - \phi(\boldsymbol{x}))^2\rangle$
- Conditional mean of line segments: $\langle(\Delta\phi)^2|\ell\rangle$

Structure Function Analysis

nth moment of $\Delta\phi$ can be written as

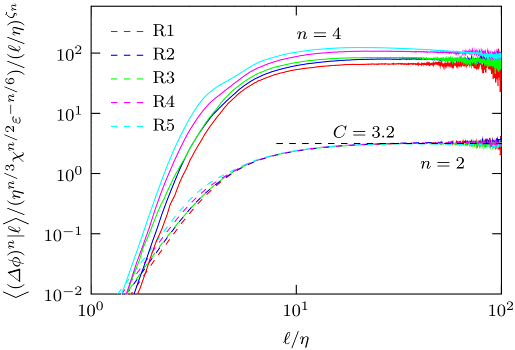
$$\langle(\Delta\phi)^n\rangle = F_n(\mathrm{Re}_r, \mathrm{Pe})\langle\chi\rangle^{n/2}\langle\varepsilon\rangle^{-n/6}r^{n/3}$$

Scaling of nth order structure function in the inertial range: $\dfrac{\langle(\Delta\phi)^n\rangle}{\langle\chi\rangle^{n/2}\langle\varepsilon\rangle^{-n/6}r^{n/3}} = c_n$



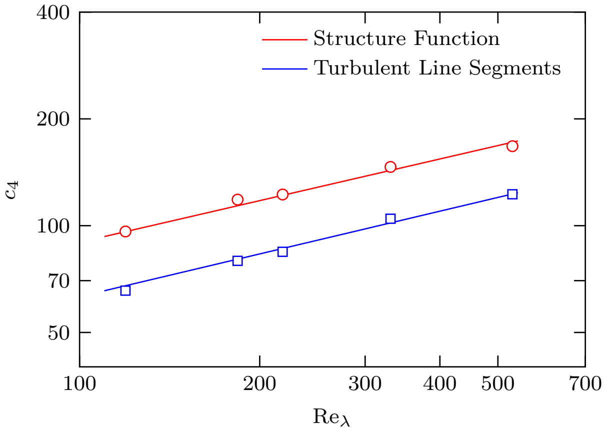
Conditional Mean of Line Segments

Scaling of nth order conditional mean in the inertial range: $\frac{\langle(\Delta\phi)^n|\ell\rangle}{\langle\chi\rangle^{n/2}\langle\varepsilon\rangle^{-n/6}r^{n/3}} = c_n$



- **c_2 is quasi-universal, c_4 is not universal**

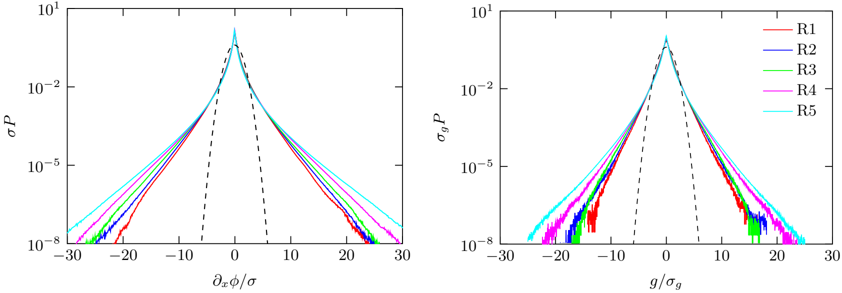
Scaling of c_4 with Reynolds number



- For structure function: $c_4 = 15\,\mathrm{Re}_\lambda^{0.4}$
- For line segments: $c_4 = 10\,\mathrm{Re}_\lambda^{0.4}$

Higher Order Statistics

Normalized marginal pdf of local gradient $\partial_x \phi$ and $g = \Delta\phi/\ell$

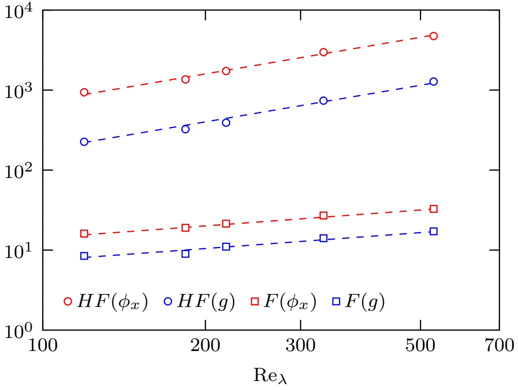


- **stretched exponential tails**, more stretched with increasing Reynolds number
- strong deviations from Gaussianity
- $P_{\partial_x\phi}(x) = c\exp(-\alpha x^\beta)$
- $\Rightarrow P(\partial_x\phi)$ has longer tails than $P(g)$

Higher Order Statistics

Scaling of Flatness and Hyperflatness with Reynolds number

- Flatness: $F(\phi_x) = \frac{\langle \phi_x^4 \rangle}{\langle \phi_x^2 \rangle^2}$ and $F(g) = \frac{\langle g^4 \rangle}{\langle g^2 \rangle^2}$
- Hyper-flatness: $HF(\phi_x) = \frac{\langle \phi_x^6 \rangle}{\langle \phi_x^2 \rangle^3}$ and $F(g) = \frac{\langle g^6 \rangle}{\langle g^2 \rangle^3}$



- $F_{\phi_x} = 1.42\,\mathrm{Re}_\lambda^{0.5}$ and $HF_{\phi_x} = 3.6\,\mathrm{Re}_\lambda^{1.15}$
- $F_{g_x} = 0.74\,\mathrm{Re}_\lambda^{0.5}$ and $HF_{g_x} = 0.91\,\mathrm{Re}_\lambda^{1.15}$

Scale Similarity of Local and Mean Gradients

Ratio of the nth order moments of ϕ_x and g

$$c_n = \frac{\langle |\phi_x|^n \rangle}{\langle |g|^n \rangle} \geq 1 \tag{1}$$

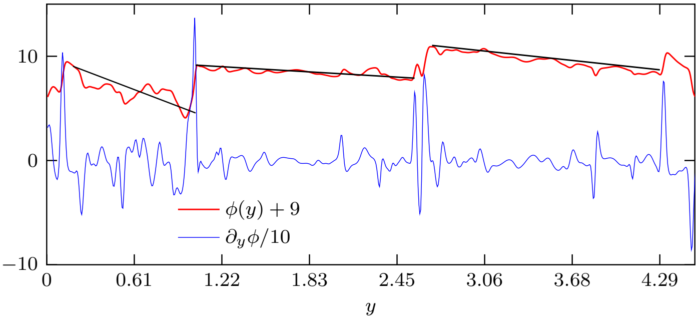
$$\frac{\langle \phi_x^{2n} \rangle}{\langle g^{2n} \rangle} = \frac{\langle (\phi_x/\sigma_{\phi_x})^{2n} \rangle}{\langle (g/\sigma_{\phi_g})^{2n} \rangle} \left(\frac{\langle \phi_x^2 \rangle}{\langle g^2 \rangle} \right)^n \propto \frac{\mathrm{Re}_\lambda^{m_{\phi_x}(2n)}}{\mathrm{Re}_\lambda^{m_g(2n)}} c_2^n \neq f(\mathrm{Re}_\lambda)\,, \tag{2}$$

Re_λ	119	184	215	331	529
c_1	1.00	1.00	1.00	1.00	1.00
c_2	1.51	1.52	1.53	1.52	1.52
c_3	2.72	2.76	2.82	2.84	2.82
c_4	5.23	5.35	5.58	5.56	5.46
c_5	10.22	10.44	11.26	11.03	10.49
c_6	19.88	20.07	22.82	21.23	19.50

$\Rightarrow$ Scale similarity

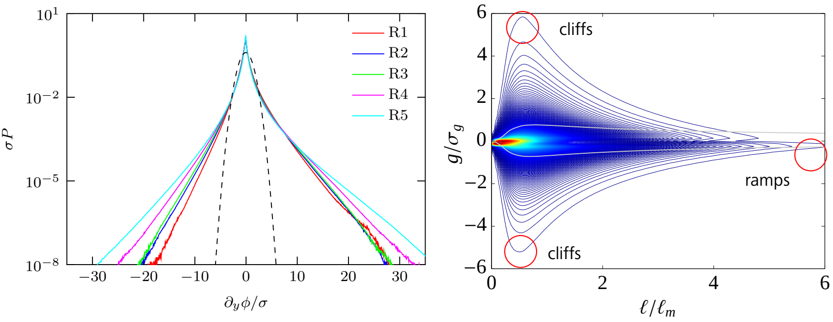
Cliff-Ramp Structures

- Skewness in scalar derivative
- Consider line segments in direction of mean scalar gradient

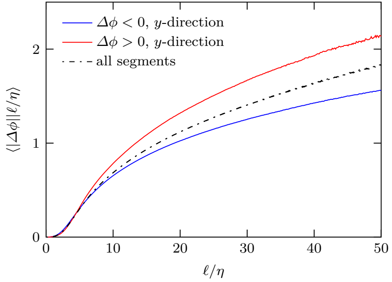


Cliff-Ramp Structures

- Scalar field exhibits skewness
- Consider line segments in direction of mean scalar gradient



Cliff-Ramp Structures



$$\langle|\Delta\phi|\,|\Delta\phi > 0\rangle \geq \langle|\Delta\phi|\,|\Delta\phi < 0\rangle \quad (3)$$

$$\frac{1}{N^+}\int_0^\infty |\Delta\phi|P(\Delta\phi)\mathrm{d}(\Delta\phi) \geq \frac{1}{N^-}\int_{-\infty}^0 |\Delta\phi|P(\Delta\phi)\mathrm{d}(\Delta\phi)\,, \quad (4)$$

$$\frac{N^-}{N^+} \geq \frac{\int_0^\infty |\Delta\phi|P(\Delta\phi)\mathrm{d}(\Delta\phi)}{\int_{-\infty}^0 |\Delta\phi|P(\Delta\phi)\mathrm{d}(\Delta\phi)} = 1 \quad (5)$$

$$\boxed{\frac{\ell_m^-}{\ell_m^+} \geq 1} \quad (6)$$

Re_λ	88	119	184	215	331	529
ℓ_m^-/ℓ_m^+	1.45	1.39	1.31	1.26	1.24	1.14
$S(\phi_y)$	1.65	1.73	1.60	1.55	1.56	1.36

- ℓ_m^-/ℓ_m^+ tends to unity for $Re_\lambda \to \infty$
- Ratio ℓ_m^-/ℓ_m^+ can be understood as surrogate for gradient skewness

Summary and Conclusion

- We proposed a **decomposition of the turbulent field** based on minimal/maxima points
- Mean length ℓ_m scales with Kolmogorov lenght η
- **Scale similarity** between the moments of ϕ_x and g
- Line Segments helps to understand scalar **skewness**