The Universal Euler Equation in Einstein Real Gravity

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Abstract

Making use of the Bianchi identity together with the Yilmaz exponential parametrization of General Relativity with the so-called tensor potentials describing the physical gravitational field, a universal hydrodynamical Euler equation is derived for a general relativistic fluid. This novel relationship remains valid in the presence of gravitational waves, as it takes into account the full tensorial character of the gravitational field. As a special case, we recover the general hydrodynamical evolution formula for the single component newtonian gravitational field of a relativistic fluid. An important application is the re-analysis of perturbative Jeans' theory of gravitational instability and the study of sound waves in the presence of a gravitational field and their interaction with gravitational waves.

1 Einstein Real Gravity Theory

Einstein General Relativity (GR) describes the laws of nature as seen from general reference frames, whether inertial or not. The equivalence principle assures us that the general motion of particles or material bodies under (loosely called) gravitational forces are equivalently described by observers in non inertial reference frames. But these loosely called gravitational forces covered by the equivalence principle are known not to be restricted to real (pure or actual) gravity. The equivalence principle in fact unifies the real gravitational force with other forces known as inertial forces such as the centrifugal and Coriolis forces of classical mechanics. All of these forces are traditionnally called gravitational forces by general relativists.

The real gravitational force is very different from the other inertial forces. Unlike the latters, the real gravitational force vanishes at infinity [1, 2, 3, 4, 5]. Furthermore, unlike the others, it cannot be removed globally by a change of reference frame. The other inertial forces do disappear when going to an inertial reference frame (cartesian or *natural* coordinates [1, 2, 3, 4, 5]). The reason why the real gravitational force connot be made to disappear is because it *curves spacetime* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The other forces do not, although they remain unified with the real gravitational force through general relativity and the equivalence principle.

One realizes that the equivalence principle is too wide a principle, englobing too many physical forces. The description of real gravity is then cluttered by these unwanted forces which complicate unnecessarily the dynamics. It thus seems reasonable to split general relativity into two parts [1, 2, 3, 4, 5, 6, 7, 8], one covering the flat spacetime inertial forces which should be incorporated into a generalization of special relativity, and another one dealing exclusively with curved spacetime phenomena, *i.e.* the real gravitational force which we call Einstein Real Gravity (RG).

Such a linear separation of forces can only be accomplished with the aid of the Yilmaz exponential parametrization of the metric tensor [1, 2, 3, 4, 5, 6, 7, 8, 9], written in terms of so-called *tensor potentials* $\phi_i^{\ j}$ as follows,

$$g_{ij} \;=\; (e^{2(\phi\hat{I}-2\hat{\phi})})_i^{\ k}\,\eta_{kj} \;=\; \psi_i^{\ k}[\hat{\Sigma}]\,\eta_{kj}\;, \tag{1.1}$$

$$\psi_i^{\ k}[\hat{\Sigma}] \;\equiv\; (e^{-4\hat{\Sigma}})_i^{\ k}\;, \tag{1.2}$$

$$(\hat{\Sigma})_i^{\ j} \;=\; \bar{\phi}_i^{\ j} \;\equiv\; \phi_i^{\ j} - \frac{1}{2}\delta_i^{\ j}\phi\;, \tag{1.3}$$

in which $\eta_{ij} = (-1, 1, 1, 1)$ is the flat spacetime metric in cartesian coordinates (which defines a galilean reference frame [10]) and where $(\hat{\phi})_i^j = \phi_i^j$ and $(\hat{I})_i^j = \delta_i^j$ with trace $\phi \equiv \phi_i^{\ i} = -\bar{\phi}$.

As discussed in previous works [1, 2, 3, 4, 5, 6, 7, 8], the Yilmaz tensor potentials ϕ_i^j are related *linearly* to Newton's gravitational potential, unlike the traditional metric representation of GR. To lowest order in Newton's constant, the traditional metric representation is only the first order term of the exponential parametrization representation and so constitutes only a weak field description. Note that both representations, to this point, equally agree with all observational and experimental tests of weak field general relativity [9].

Unlike the traditional weak field metric representation, the strong field exponential parametrization displays no event horizon. Therefore black holes do not exist in this parametrization and no singularity ever develops [1, 2, 3, 4, 5, 6, 7, 8, 9]. The universe is everywhere regular, in complete agreement with basic observations.

Let us now take into consideration the existence of the two physically independent and separate gravitational potentials representing respectively the pure or *real gravitational potentials* φ_i^j and the coordinates or *inertial potentials* χ_i^j expressed in the following manner [1, 2, 3, 4, 5, 6, 7, 8],

$$\hat{\Sigma} \;=\; \hat{\Phi} + \hat{\Omega} \quad ; \quad \phi_i^{\ j} \;=\; \varphi_i^{\ j} + \chi_i^{\ j} \quad , \tag{1.4}$$

leading to the following metric tensor,

$$g_{ij} \;=\; \psi_i^{\ k}[\hat{\Sigma}]\, \eta_{kj} \;=\; \psi_i^{\ k}[\hat{\Phi} + \hat{\Omega}]\, \eta_{kj} \;, \tag{1.5}$$

with the definitions,

$$(\hat{\Phi})_i^{\ j} \;=\; \bar{\varphi}_i^{\ j} \;\equiv\; \varphi_i^{\ j} \;-\; \frac{1}{2}\delta_i^{\ j}\varphi \;, \tag{1.6}$$

$$(\hat{\Omega})_i^{\ j} \;=\; \bar{\chi}_i^{\ j} \;\equiv\; \chi_i^{\ j} \;-\; \frac{1}{2}\delta_i^{\ j}\chi \;. \tag{1.7}$$

In the case of *diagonal* metrics such as the central symmetric isotropic spacetime [7, 8], the gravitational and inertial potentials are diagonal as well and equation (1.5) for the tensor potentials $\phi_i^{\ j}$ cleanly factorizes as follows,

$$g_{ij} \;=\; \psi_i^{\ k}[\hat{\Sigma}]\, \eta_{kj} \;=\; \psi_i^{\ k}[\hat{\Phi} + \hat{\Omega}]\, \eta_{kj} \;=\; \psi_i^{\ k}[\hat{\Phi}]\, \eta_{kj}[\hat{\Omega}], \tag{1.8}$$

where we defined,

$$\eta_{kj}[\hat{\Omega}] \;\equiv\; \psi_k^{\ i}[\hat{\Omega}]\, \eta_{ij} \quad , \tag{1.9}$$

which is a coordinate transformation from inertial (cartesian) to non inertial (spherical, cylindrical, etc.) flat spacetime coordinates.

Separation of the Christoffel symbols

Yilmaz exponential parametrization of the metric (1.1)-(1.7) now leads to the following general expression for the Christoffel symbols in terms of the tensor potentials $\bar{\phi}_i^{\ j}$,

$$\Gamma^i_{\ kl}[\hat{\Sigma}] \;=\; -4\partial_{(l}\bar{\phi}_{k)}^{\ i} \;+\; 2g^{im}\partial_m\bar{\phi}_{(l}^{\ j}g_{jk)} \;. \tag{1.10}$$

Recalling the separation (1.4)-(1.7) in terms of the real gravitational potentials $\bar{\varphi}_i^j$ and the inertial potentials $\bar{\chi}_i^j$, we get the following linear separation of the Christoffel symbols,

$$\Gamma^i_{\ kl}[\hat{\Sigma}] \;=\; \Delta^i_{\ kl}[\hat{\Phi}] + S^i_{\ kl}[\hat{\Omega}] \;, \tag{1.11}$$

in which we defined the *Gravitational symbols* $\Delta^i_{\ kl}$ and *Inertial symbols* $S^i_{\ kl}$ as follows,

$$\Delta^i_{\ kl}[\hat{\Phi}] \;\equiv\; -4\partial_{(l}\bar{\varphi}_{k)}^{\ i} \;+\; 2g^{im}\partial_m\bar{\varphi}_{(l}^{\ j}g_{jk)} \;, \tag{1.12}$$

and,

$$S^i_{\ kl}[\hat{\Omega}] \;\equiv\; -4\partial_{(l}\bar{\chi}_{k)}^{\ i} \;+\; 2g^{im}\partial_m\bar{\chi}_{(l}^{\ j}g_{jk)} \;. \tag{1.13}$$

For general and inertial (cartesian) coordinates, we have respectively,

$$\Gamma^i_{\ kl} = S^i_{\ kl} + \Delta^i_{\ kl} \;, \tag{1.14}$$

$$\Gamma'^{i}_{\ kl} = S'^{i}_{\ kl} + \Delta'^{i}_{\ kl} \ , \tag{1.15}$$

with the following transformation formula,

$$\Gamma^{i}_{\ kl} = \Gamma'^{s}_{\ rp} \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial x^i}{\partial x'^s} \frac{\partial^2 x'^s}{\partial x^k \partial x^l} \ , \tag{1.16}$$

and so,

$$S^{i}_{\ kl} + \Delta^{i}_{\ kl} = (S'^{s}_{\ rp} + \Delta'^{s}_{\ rp}) \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial x^i}{\partial x'^s} \frac{\partial^2 x'^s}{\partial x^k \partial x^l} \ . \tag{1.17}$$

Now the Inertial symbols are pure coordinate transformations from inertial (cartesian) x'^s to general x^k coordinates,

$$S^{i}_{\ kl} = \frac{\partial x^i}{\partial x'^s} \frac{\partial^2 x'^s}{\partial x^k \partial x^l} \ , \tag{1.18}$$

which leads directly to $S'^{s}_{\ rp} = 0$ in inertial (cartesian) coordinates. This is so because the only transformations between inertial coordinates are linear Lorentz transformations, which is the statement of vanishing inertial (non-tensor) forces in inertial coordinate systems.

Now since $S'^{s}_{\ rp} = 0$, eqs. (1.17)-(1.18) further imply that the Gravitational symbols $\Delta^{i}_{\ kl}$ are themselves pure tensors,

$$\Delta^{i}_{\ kl} = \Delta'^{s}_{\ rp} \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^p}{\partial x^l} \ , \tag{1.19}$$

which means that *the real gravitational force is a pure tensor force* [1, 2, 3, 4, 5, 6, 7, 8].

2 The Freud and Einstein-Pauli pseudo-tensors

It is well known that the Einstein tensor $G_i^{\ k}$ can be decomposed [1, 2, 3, 4, 5, 6, 7, 8, 9] in terms of the Freud $F_i^{\ k}$ and Einstein-Pauli $E_i^{\ k}$ pseudo-tensors as follows,

$$\sqrt{-g} G_i^{\ k} \ = \ \sqrt{-g}(F_i^{\ k} \ - \ E_i^{\ k}) \ . \tag{2.1}$$

The Einstein-Pauli pseudo-tensor is given as [1, 2, 3, 4, 5, 6, 7, 8, 9, 10],

$$\begin{aligned} \sqrt{-g}\, E_j^{\ k} \ &= \ \sqrt{-g}\left(W_j^{\ k} - \frac{1}{2}\, \delta_j^{\ k}\, W \right) \\ \sqrt{-g}\, E \ &\equiv \ \sqrt{-g}\, E_j^{\ k} \delta_k^{\ j} \ = \ -\sqrt{-g}\, W \ , \end{aligned} \tag{2.2}$$

where,

$$\begin{aligned} \sqrt{-g}\, W_j^{\ k} \ &\equiv \ -8\, \mathbf{g}^{ks}\left(\partial_j \phi_l^{\ m} \partial_m \phi_s^{\ l} - \frac{1}{2} \partial_j \phi_l^{\ m} \partial_s \phi_m^{\ l} + \frac{1}{4} \partial_j \phi \partial_s \phi \right) \\ \frac{1}{2}\sqrt{-g}\, W \ &= \ -4\, \mathbf{g}^{rs}\left(\partial_l \phi_r^{\ m} \partial_m \phi_s^{\ l} - \frac{1}{2} \partial_r \phi_l^{\ m} \partial_s \phi_m^{\ l} + \frac{1}{4} \partial_r \phi \partial_s \phi \right) \ , \end{aligned} \tag{2.3}$$

and in which we defined the **rescaled metric tensor**,

$$\mathbf{g}^{ij} \equiv \sqrt{-g} g^{ij} \ \ ; \ \ \mathbf{g}_{ij} \equiv \frac{1}{\sqrt{-g}} g_{ij} \ . \tag{2.4}$$

The Freud pseudo-tensor on the other hand can be expressed in the following manner [1, 2, 3, 4, 5, 6, 7, 8, 9],

$$\sqrt{-g}F_j^{\ k} \ \equiv\ \partial_l(\sqrt{-g}B_j^{\ kl})\ , \tag{2.5}$$

with the antisymmetric super-potential $B_j^{\ kl}$ given by [1, 2, 3, 4, 5, 6, 7, 8, 9],

$$\begin{aligned}\sqrt{-g}B_j^{\ kl} = \sqrt{-g}B_j^{\ [kl]} \ &=\ -\frac{1}{2}\sqrt{-g}\,\big[\,\delta_j^{\ k}(g^{rs}\Gamma^l_{\ rs} - g^{lr}\Gamma^s_{\ rs}) \\ &\quad +\ \delta_j^{\ l}(g^{kr}\Gamma^s_{\ rs} - g^{rs}\Gamma^k_{\ rs}) \\ &\quad +\ (g^{lr}\Gamma^k_{\ jr} - g^{kr}\Gamma^l_{\ jr})\big]\ ,\end{aligned} \tag{2.6}$$

with the property $\partial_k\partial_l(\sqrt{-g}B_j^{\ kl}) = 0$. The antisymmetry of the super-potential in turn leads directly to the so-called **Freud identity** [1, 2, 3, 4, 5, 6, 7, 8, 9],

$$\partial_k(\sqrt{-g}F_j^{\ k})\ =\ 0\ , \tag{2.7}$$

which is related to energy-momentum conservation in general relativity.

Einstein general relativity is then described as follows [1, 2, 3, 4, 5, 6, 7, 8],

$$G_j^{\ k} = \frac{8\pi k}{c^4}\,\tau_j^{(\mathrm{E})k} \quad ; \quad E_j^{\ k} = \frac{8\pi k}{c^4}\,\tilde{t}_j^{\ (\mathrm{E})k} \quad ; \quad F_j^{\ k} = \frac{8\pi k}{c^4}\,(\tau_j^{(\mathrm{E})k} + \tilde{t}_j^{\ (\mathrm{E})k})\ , \tag{2.8}$$

with matter energy-momentum tensor $\tau_j^{(\mathrm{E})k}$ and gravitational energy-momentum pseudo-tensor $\tilde{t}_j^{\ (\mathrm{E})k}$.

Finally, the Einstein tensor $G_i^{\ k}$ satisfies the important **Bianchi identity**,

$$D_kG_j^{\ k}\ =\ D_k(F_j^{\ k} - E_j^{\ k})\ =\ 0\ , \tag{2.9}$$

3 The Universal Matter Dynamics Equation

As shown in our previous work [2], the hydrostatic equilibrium equation can be generalized to the hydrodynamic case with a spacetime-dependent newtonian gravitational field. Here we generalize the dynamics even further [1] by taking into account the full tensorial characteristic of the so-called tensor potentials describing the full gravitational field. Such a generalization turns out to be one of the easiest formula to derive from general relativity.

Starting from the Bianchi identity (2.9) as well as the rule for the covariant derivative of *symmetric* tensors [10], we write immediately,

$$D_kG_j^{\ k}\ =\ \frac{1}{\sqrt{-g}}\,\partial_k(\sqrt{-g}\,G_j^{\ k}) - \frac{1}{2}\,\partial_j g_{kl}\,G^{kl}\ =\ 0\ . \tag{3.1}$$

We then find,

$$\partial_k(\sqrt{-g}\,G_j^{\ k})\ =\ \frac{1}{2}\,(g^{km}\partial_j g_{kl})\,\sqrt{-g}\,G_m^{\ \ l}\ . \tag{3.2}$$

Recalling the exponential parametrization (1.1)-(1.3) of the metric tensor, we get,

$$g^{km}\partial_j g_{kl}\ =\ -4\,\partial_j\bar{\phi}_l^{\ m}\ . \tag{3.3}$$

Insertion into eq. (3.2) yields,

$$\begin{aligned}\partial_k(\sqrt{-g}\,G_j^{\;k}) &= -2\,(\partial_j\bar{\phi}_l^{\;m})\,\sqrt{-g}\,G_m^{\;\;l}\\ &= -2\,(\partial_j\phi_l^{\;m})\,\sqrt{-g}\,R_m^{\;\;l}\;, \end{aligned}\qquad (3.4)$$

the second line of which being derived from the definition (1.3) involving the gravitational tensor potential field $\phi_l^{\;m}$. This is the field which reduces to the newtonian gravitational field.

Recalling the Einstein field equation (2.8), we finally arrive at the following forms for the **universal matter dynamics equation** (UMDE),

$$\begin{aligned}\partial_k(\sqrt{-g}\,\tau_j^{(\mathrm{E})k}) &= -2\,(\partial_j\bar{\phi}_l^{\;m})\,\sqrt{-g}\,\tau_m^{(\mathrm{E})l}\\ &= -2\,(\partial_j\phi_l^{\;m})\,\sqrt{-g}\left(\tau_m^{(\mathrm{E})l}-\frac{1}{2}\delta_m^{\;l}\tau^{(\mathrm{E})}\right), \end{aligned}\qquad (3.5)$$

with the trace of the Einstein energy-momentum tensor defined as $\tau^{(\mathrm{E})}\equiv\tau_l^{(\mathrm{E})l}$.

We note as well the following formula which can be obtained from the Freud identity (2.7) with eq. (2.8) for the Freud pseudo-tensor,

$$\partial_k(\sqrt{-g}\,\tau_j^{(\mathrm{E})k}) \;=\; -\partial_k(\sqrt{-g}\,\tilde{t}_j^{(\mathrm{E})k})\;. \qquad (3.6)$$

The UMDE (3.5) is valid in *any* coordinate system, whether curvilinear or cartesian and for *any* matter energy-momentum tensor. However, as discussed previously [1, 2, 3, 4, 5], the use of curvilinear coordinates necessarily creates additional non-tensor forces in the description of the system, the so-called inertial forces, which clutter the dynamics of the real gravitational field. A clean view of real gravitational physics is obtained by constraining the theory to a description in terms of cartesian (natural) coordinates. In such an inertial reference frame, the inertial potentials and inertial forces vanish. In such a reference frame, the tensor potentials are given by the real gravitational potentials ($\phi_i^j=\varphi_i^j$) and the Christoffel symbols given by the Gravitational symbols ($\Gamma^i{}_{kl}=\Delta^i{}_{kl}$) which are true tensors [1, 2, 3, 4, 5, 6, 7, 8]. Einstein general relativity then becomes an entirely *localized* (no pseudo-tensors) affine tensor theory. In such a frame, (3.5) becomes,

$$\begin{aligned}\partial_k(\sqrt{-g}\,\tau_j^{(\mathrm{E})k}) &= -2\,(\partial_j\bar{\varphi}_l^{\;m})\,\sqrt{-g}\,\tau_m^{(\mathrm{E})l}\\ &= -2\,(\partial_j\varphi_l^{\;m})\,\sqrt{-g}\left(\tau_m^{(\mathrm{E})l}-\frac{1}{2}\delta_m^{\;l}\tau^{(\mathrm{E})}\right)\\ &= -\partial_k(\sqrt{-g}\,t_j^{(\mathrm{E})k})\;, \end{aligned}\qquad (3.7)$$

with $t_j^{(\mathrm{E})k}$ the gravitational energy-momentum *affine tensor* expressed as,

$$\begin{aligned}\sqrt{-g}\,t_j^{(\mathrm{E})k} \;=\; \frac{c^4}{8\pi k}\Big[&-8\,\mathbf{g}^{ks}\left(\partial_j\varphi_l^{\;m}\partial_m\varphi_s^{\;l}-\frac{1}{2}\partial_j\varphi_l^{\;m}\partial_s\varphi_m^{\;l}+\frac{1}{4}\partial_j\varphi\partial_s\varphi\right)\\ &+4\,\delta_j^k\,\mathbf{g}^{rs}\left(\partial_l\varphi_r^{\;m}\partial_m\varphi_s^{\;l}-\frac{1}{2}\partial_r\varphi_l^{\;m}\partial_s\varphi_m^{\;l}+\frac{1}{4}\partial_r\varphi\partial_s\varphi\right)\Big]\;. \end{aligned}\qquad (3.8)$$

The Newtonian Potential

Let us now check that we recover the results of our previous works for a newtonian potential [2, 3, 4]. Assuming Newton's potential Φ to be spacetime dependent, the physical tensor potential $\varphi_j^{\ k}$ is now given as follows [2, 3, 4],

$$\varphi_j^{\ k}(x^0,\vec{x}) = -\frac{\Phi(x^0,\vec{x})}{c^2}\,\delta_j^{\ 0}\delta_0^{\ k}\ , \tag{3.9}$$

which leads to the following expression for the rescaled metric tensor [2, 3, 4],

$$\mathbf{g}^{00} = -e^{-4\Phi/c^2}\ \ ;\ \ \mathbf{g}^{\alpha\beta} = \eta^{\alpha\beta}\ \ ;\ \ \sqrt{-g} = e^{-2\Phi/c^2}\ . \tag{3.10}$$

Inserting the newtonian value (3.9) for the tensor potential field into the second line expression for the UMDE (3.7) we immediately recover the formula,

$$\partial_k(\sqrt{-g}\,\tau_j^{(\mathrm{E})k}) \ \ = \ \ \sqrt{-g}\,(\tau_0^{(\mathrm{E})0} - \tau_\alpha^{(\mathrm{E})\alpha})\,\partial_j\left(\frac{\Phi}{c^2}\right)\ , \tag{3.11}$$

which is called the **General Hydrodynamical Evolution Equation** (HDEE) of Einstein Real Gravity [2] when the Einstein energy-momentum tensor $\tau_j^{(\mathrm{E})k}$ describes a general relativistic fluid.

For a perfect fluid in a time-independent gravitational field ($\partial_0\Phi = 0$), we recover the usual **Hydrostatic Equilibrium Equation** (HSEE) [3, 4],

$$-\,\partial_\alpha(\sqrt{-g}\,p) \ \ = \ \ \sqrt{-g}\,(\,e + 3p\,)\,\partial_\alpha\left(\frac{\Phi}{c^2}\right)\ . \tag{3.12}$$

This relationship has been shown to lead to a modified and *non singular* Tolman-Oppenheimer-Volkoff equation for star equilibrium [3, 4].

Note that the Tolman terms $(e + 3p)$ in (3.12) and $(\tau_0^{(\mathrm{E})0} - \tau_\alpha^{(\mathrm{E})\alpha})$ in (3.11) originate from the term $(\tau_m^{(\mathrm{E})l} - \frac{1}{2}\delta_m^{\ l}\tau^{(\mathrm{E})})$ in the second line of eqs. (3.7) or (3.5), *i.e.* the Ricci curvature term $R_m^{\ l}$ in the second line of eq. (3.4).

A closed form analytical solution of eq. (3.12) was found [2], with gravitational potential given by ($r > 0$),

$$\frac{\Phi(r)}{c^2} \ \ = \ \ -\frac{1}{3}\ln\left(\frac{r_0}{r}\right) - \frac{R_0}{r}\ \ , \tag{3.13}$$

with free radii parameters r_0 and R_0 determined from certain boundary conditions. The second term with R_0 in eq. (3.13) is the homogeneous part given by the solution of the Laplace equation $\vec{\nabla}^2\Phi = 0$.

We further found [2] ($r > 0$),

$$e(r) \ = \ \frac{c^4\,e^{-2R_0/r}}{24\pi k r_0^2}\left(\frac{r_0}{r}\right)^{4/3}\ ;\ \ p(r) \ = \ \frac{c^4\,e^{-2R_0/r}}{72\pi k r_0^2}\left(\frac{r_0}{r}\right)^{4/3}\ , \tag{3.14}$$

which, for finite positive R_0, allow the density and pressure to vanish in the limit $r \to 0$.

On the other hand, the *observable* energy and pressure densities are given by the product of the proper densities with the factor $\sqrt{-g_{00}(r)}$ since the invariant spatial volume element in the reference frame of the calculation (the observer) is given by $\frac{\sqrt{-g}}{\sqrt{-g_{00}}}\, dV$ for a static spacetime ($g_{0\alpha} = 0$). Therefore we find ($r > 0$),

$$\sqrt{-g_{00}(r)}\, e(r) = \frac{c^4\, e^{-3R_0/r}}{24\pi k r_0^2}\left(\frac{r_0}{r}\right) \;\; ; \;\; \sqrt{-g_{00}(r)}\, p(r) = \frac{c^4\, e^{-3R_0/r}}{72\pi k r_0^2}\left(\frac{r_0}{r}\right) , \qquad (3.15)$$

which again vanish in the limit $r \to 0$.

The results of eq. (3.14) in terms of the proper energy density $e(r)$ and proper pressure density $p(r)$ are to be compared with the high-density neutron star results [11, 12] from the old Tolman-Oppenheimer-Volkoff hydrostatic equilibrium formula for the Schwarzschild metric,

$$e(r) \;=\; \frac{3c^4}{56\pi k r^2} \;\; ; \;\; p(r) \;=\; \frac{c^4}{56\pi k r^2} \;\; , \qquad (3.16)$$

which are infinite in the limit $r \to 0$.

4 The Euler (Navier-Stokes) Equation

Equation (3.7) is an important universal matter dynamics relationship. We emphasize its universality because it can describe all possible physical situations of astrophysical matter source evolution, including viscosity, heat transport at finite temperature, spacetime-dependent internal energy and pressure, magneto-hydrodynamical and radiative effects, all characteristics of imperfect fluids, as well as gravitational waves production, because of the full tensorial character of the tensor potentials included in the formula.

In astrophysics, the Einstein energy-momentum tensor to treat in a unified way all of these cases is given as follows,

$$\begin{aligned} \tau_j^{(\mathrm{E})k} &= [\, e\, u_j u^k + p_j^{\;k}\,] \\ p_j^{\;k} &\equiv p\, \Delta_j^{\;k} + \tau_j^{(\mathrm{vsc})k} + \tau_j^{(\mathrm{em})k} \\ \Delta_j^{\;k} &\equiv u_j u^k + \delta_j^{\;k} \;\; ; \;\; u^j \Delta_j^{\;k} = 0 \;\; , \end{aligned} \qquad (4.1)$$

where (e, p) are the internal energy density and normal pressure respectively, while $\tau_j^{(\mathrm{vsc})k}$ and $\tau_j^{(\mathrm{em})k}$ stand for the viscosity and electromagnetic stress-energy tensors given respectively as follows,

$$\begin{aligned} \tau_j^{(\mathrm{vsc})k} &= -\eta\, (\, \Delta^{kl} D_l u_j + \Delta_j^{\;l} D_l u^k\,) + (\zeta - \frac{2}{3}\eta)\, \Delta_j^{\;k} D_l u^l \; , \\ \tau_j^{(\mathrm{em})k} &= \frac{1}{4\pi}\, (\, F_{jl} F^{kl} - \frac{1}{4}\delta_j^k F_{lm} F^{lm}\,) \; , \end{aligned} \qquad (4.2)$$

with viscosity coefficients (η, ζ) and electromagnetic tensor $F_{ik} \equiv \partial_i A_k - \partial_k A_i$.

The Einstein energy-momentum tensor (4.1) can be re-written as,

$$\begin{aligned} \tau_j^{(\mathrm{E})k} &= [\, w\, u_j u^k + p\, \delta_j^{\;k} + \tau_j^{(\mathrm{vsc})k} + \tau_j^{(\mathrm{em})k}\,] \\ w &\equiv e + p \;\; , \end{aligned} \qquad (4.3)$$

with w the enthalpy.

The Euler equation in traditional General Relativity is derived by starting with the Bianchi identity (3.1) written in terms of the Einstein energy-momentum tensor,

$$D_k \tau_j^{(\mathrm{E})k} \; = \; 0 \; . \tag{4.4}$$

One then projects this equation to spaces parallel and perpendicular to the 4-velocity u^j. We then get trivially,

$$u^j D_k \tau_j^{(\mathrm{E})k} \; = \; 0 \; , \tag{4.5}$$

and,

$$\Delta_l^{\;j} D_k \tau_j^{(\mathrm{E})k} \; = \; 0 \; , \tag{4.6}$$

from which one derives respectively the entropy flux equation and the Euler equation *per unit mass* [13, 14],

$$D_k \sigma^k \; = \; -\nu^k D_k \frac{\mu}{T} - \frac{1}{T} \tau_j^{(\mathrm{vsc})k} D_k u^j + \frac{1}{T} u^j D_k \tau_j^{(\mathrm{em})k} \; , \tag{4.7}$$

and,

$$w\, u^k D_k u_j \; = \; -D_k (p \delta_j^{\;k} + \tau_j^{(1)k}) - u_j u^k D_l (p \delta_k^{\;l} + \tau_k^{(1)l}) \; , \tag{4.8}$$

in which $u^j \tau_j^{(\mathrm{vsc})k} = 0$ [13] and where we defined $\tau_j^{(1)k} \equiv \tau_j^{(\mathrm{vsc})k} + \tau_j^{(\mathrm{em})k}$ as well as the relativistic entropy density flux [13],

$$\sigma^k \; \equiv \; \sigma u^k - \frac{\mu}{T} \nu^k \; , \tag{4.9}$$

with σ the entropy density and ν^k a supplementary parameter explicitly given by [13],

$$\nu^k \; = \; -\frac{\chi}{c} \left(\frac{nT}{w} \right)^2 \Delta^{kl} D_l \frac{\mu}{T} \; , \tag{4.10}$$

and proportional to the coefficient of thermal conductivity χ.

Both equations (4.7) or (4.8) are to be supplemented by the continuity equation [13],

$$D_k n^k \; = \; 0 \quad ; \quad n^k \; \equiv \; n u^k + \nu^k \quad ; \quad u^j \nu_j \; = \; 0 \; , \tag{4.11}$$

with n the matter (particle number) density and n^k the 4-vector matter density flux (current), as well as an equation of state satisfying the following thermodynamical identity [13],

$$d \frac{\mu}{T} \; = \; -\left(\frac{w}{nT^2} \right) dT \; + \; \frac{1}{nT} \, dp \; . \tag{4.12}$$

The Euler Equation per Unit Volume

It is often more convenient however to make use of the Euler equation *per unit volume*, *i.e.* the equation for the conserved energy-momentum tensor (3.5), *i.e.* the Freud identity,

$$\begin{aligned} \partial_k (\sqrt{-g}\, \tau_j^{(\mathrm{E})k}) \; &= \; -\partial_k (\sqrt{-g}\, \tilde{t}_j^{(\mathrm{E})k}) \\ &= \; -2 \, (\partial_j \phi_l^{\;m}) \sqrt{-g} \left(\tau_m^{(\mathrm{E})l} \; - \; \frac{1}{2} \delta_m^{\;l} \tau^{(\mathrm{E})} \right) , \end{aligned} \tag{4.13}$$

with Tolman term $(\tau_m^{(\mathrm{E})l} - \frac{1}{2}\delta_m^{\ l}\tau^{(\mathrm{E})})$, essentially the Ricci curvature $R_m^{\ l}$.

Making use of the Einstein energy-momentum tensor of eq. (4.3), we find explicitly,

$$\begin{aligned}
&\partial_k[\sqrt{-g}\,(\,w\,u_j u^k + p\,\delta_j^{\ k} + \tau_j^{(\mathrm{vsc})k} + \tau_j^{(\mathrm{em})k}\,)] \\
&\quad = \quad -2\,(\partial_j\phi_l^{\ m})\,\sqrt{-g}\,\Big[(\,w\,u_m u^l + p\,\delta_m^{\ l} + \tau_m^{(\mathrm{vsc})l} + \tau_m^{(\mathrm{em})l}\,) \\
&\qquad\qquad -\frac{1}{2}\delta_m^l(\,-w + 4p + \tau^{(\mathrm{vsc})} + \tau^{(\mathrm{em})}\,)\Big]\ , \qquad (4.14)
\end{aligned}$$

which is a universal magneto-hydrodynamical Euler (Navier-Stokes) equation for astrophysical matter evolution (stars, galaxies, clusters, etc.). This is an entirely new form for the Euler equation owing mainly to the presence of the Tolman term, which greatly simplifies the dependence on the explicit gravitational field in the equation. This is only possible in the context of the exponential (Yilmaz) parametrization (1.1)-(1.3) of the metric tensor of Einstein General Relativity.

An important investigation avenue of the above general relativistic Navier-Stokes equation is the application of the perturbative Jeans' theory of gravitational instability (galaxy formation, etc.) [15], as well as the study of sound waves and their interactions with gravitational waves in collapsing objects.

5 Concluding Remarks

Of particular interest is the case of the so-called eternally collapsing objects (ECOs/MECOs) [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] which would involve an electromagnetic stress-energy tensor describing γ-radiation, possibly in the geometrical optics (eikonal) approximation.

We already discussed in previous works [3, 4] the case of white dwarfs and neutron stars. ECOs/MECOs on the other hand are believed to achieve the so-called Eddington balance (equilibrium) between an outward radiation flux and gravitational compression.

Let us recall the hydrostatic equation (3.12), which can be re-written as [3, 4],

$$\begin{aligned}
\partial_r(\sqrt{-g}\,p) &= -\sqrt{-g}\,[\,e(r) + 3p(r)\,]\,\partial_r\left(\frac{\Phi}{c^2}\right) \\
&= -\frac{k}{4\pi r^4}\,M_1(r)\,\partial_r M_1(r)\ \ , \qquad (5.1)
\end{aligned}$$

where the gravitational mass $M_1(r)$ at radius r is related to the Tolman density term as follows [3],

$$\begin{aligned}
\partial_r M_1(r)\,c^2 &= 4\pi r^2\sqrt{-g(r)}\,[\,e(r) + 3p(r)\,] \\
M_1(r)\,c^2 &\equiv \int_0^r 4\pi r^2 dr\sqrt{-g(r)}\,[\,e(r) + 3p(r)\,]\ \ . \qquad (5.2)
\end{aligned}$$

Now when the pressure inside the fluid is dominated by radiation pressure $p \simeq p_{rad}$ from an outward radiation flux, the luminosity L_r from the surface of the spherical fluid at radius r is given by,

$$\frac{-\partial_r(\sqrt{-g}\,p_{rad})}{\sqrt{-g}\,[\,e(r) + 3p_{rad}(r)\,]} = \partial_r\left(\frac{\Phi}{c^2}\right) = \frac{\kappa}{c^3}\,\frac{L_r}{4\pi r^2} = \frac{k}{c^2 r^2}\,M_1(r)\ \ , \qquad (5.3)$$

with constant opacity κ of the spherical stellar fluid. We then find trivially,

$$L_r = \frac{4\pi k M_1(r) c}{\kappa} \equiv L_{r,Edd} \ , \tag{5.4}$$

which is the so-called Eddington luminosity for a gravitational mass $M_1(r)$ at radius r.

Moving from hydrostatic equilibrium to the magneto-hydrodynamical Euler equation (4.14), one should be in a position to observe the effects of radiation pressure on gravitational collapse and its possible slow down toward the ECO/MECO state. This is work for the future.

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